Computational modeling of a network of excitatory stochastic spiking neurons in the mean-field limit

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Introducing the system

- N neurons

Galves, A., Löcherbach, E., 2015.

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Introducing the system

- N neurons
- $W_{ij} = 1/N \quad \forall i \neq j$

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- $\phi(x)$: spiking rate function

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- $\phi(x)$: spiking rate function
- $N \to \infty$

Galves, A., Löcherbach, E., 2015.

Introducing the System

$I \subset \mathbb{R}_+, \int_I \rho_t(r)dr$

De Masi, A., Galves, A., Löcherbach, E., Presutti, E., 2015.

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$I \subset \mathbb{R}_+, \ \int_I \rho_t(r)dr$

$$\boxed{\frac{\partial}{\partial t}\rho_t + \frac{\partial}{\partial x}(V\rho_t) = -\phi\rho_t, \quad x > 0, t > 0}$$

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$$\boxed{\int_0^\infty \rho_t(x)\mathrm{d}x = 1 \quad \forall t \geq 0}$$

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$$\rho_0(x) = u_0(x)$$

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Solving methods

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- Numerical solution: Lax-Wendroff method

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Transient Solution

$$\phi(x) = \begin{cases} 0 \text{ for } x < x_{sub} \\ \frac{x - x_{sub}}{x_{sup} - x_{sub}} \text{ for } x_{sub} \leq x < x_{sup} \\ 1 \text{ for } x \geq x_{sup} \end{cases}$$

$$\rho_0(x) = \lambda_0 \exp\left(-\lambda_0 x\right)$$

$$\lambda_0 = \lambda_0(\lambda)$$

Transient Solution

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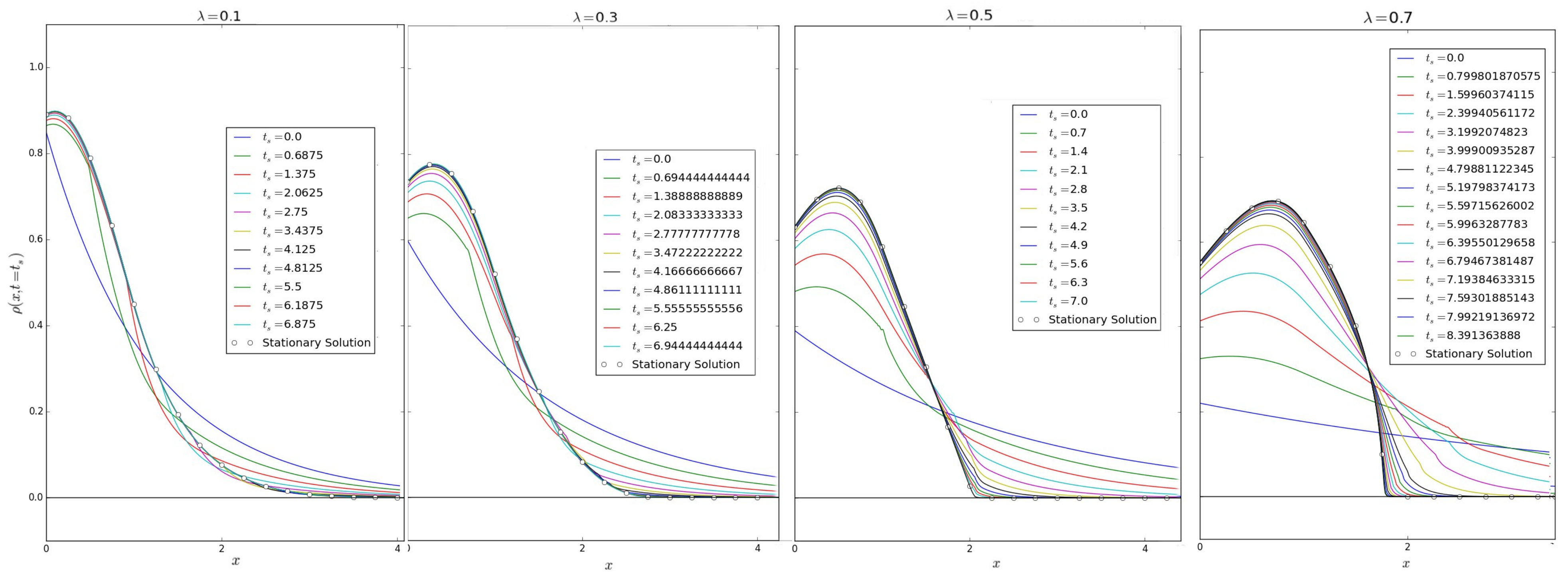
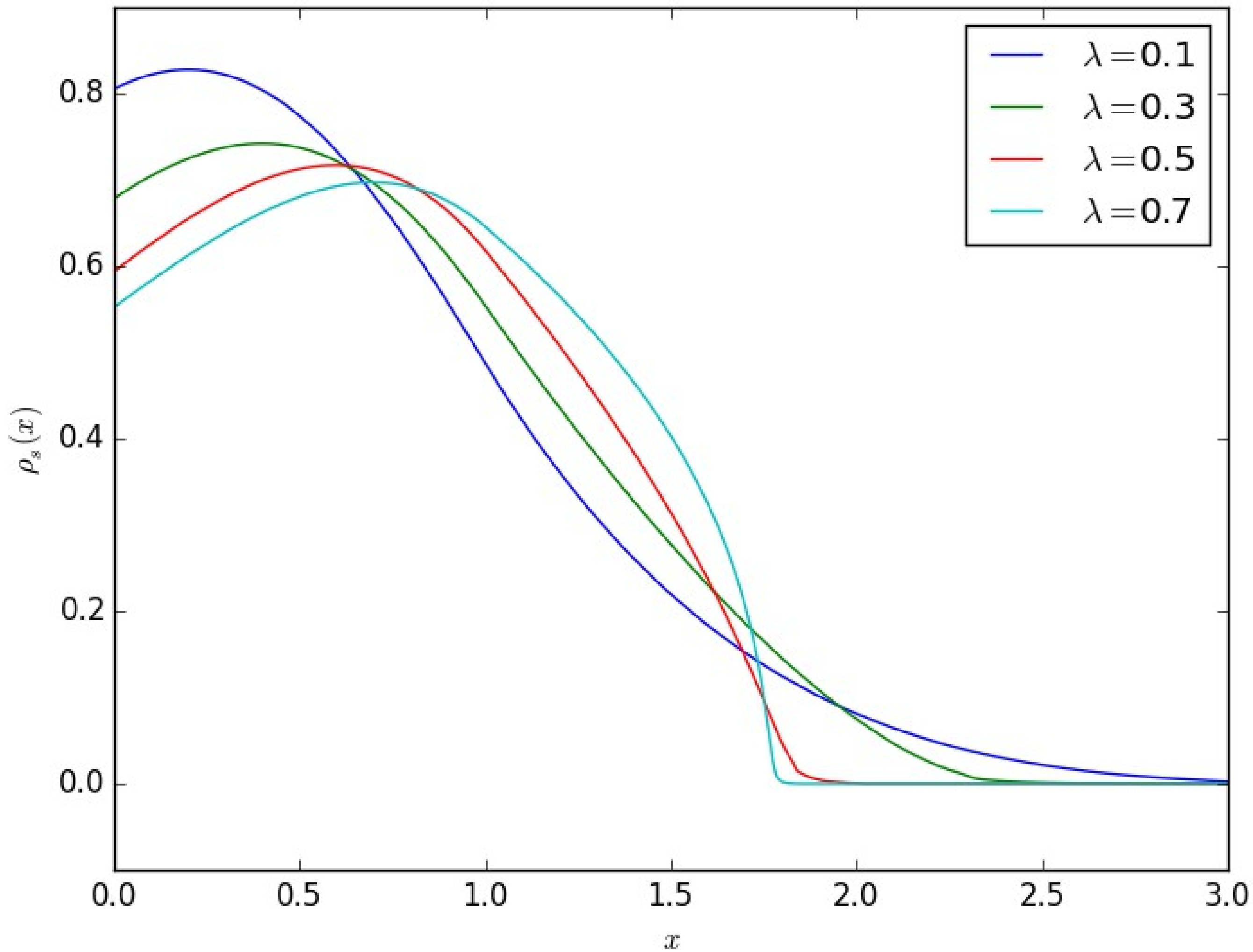


Stationary Solution



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Analytical Solution: Method of characteristics

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial}{\partial x}\big(c(x)\rho(x,t)\big) = -\phi \rho(x,t)$$

$$c(x) = -\lambda(x - x_0) + p \qquad\qquad \rho(x, t=0) = \rho_0(x)$$

$$\frac{\mathrm{d}\rho(x,t)}{\mathrm{d}t} = \frac{\partial \rho(x,t)}{\partial t} + \frac{\partial \rho(x,t)}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial \rho(x,t)}{\partial t} + c(x)\frac{\partial \rho(x,t)}{\partial x} = -\left(\phi + \frac{\mathrm{d}c(x)}{\mathrm{d}x}\right)\rho(x,t)$$

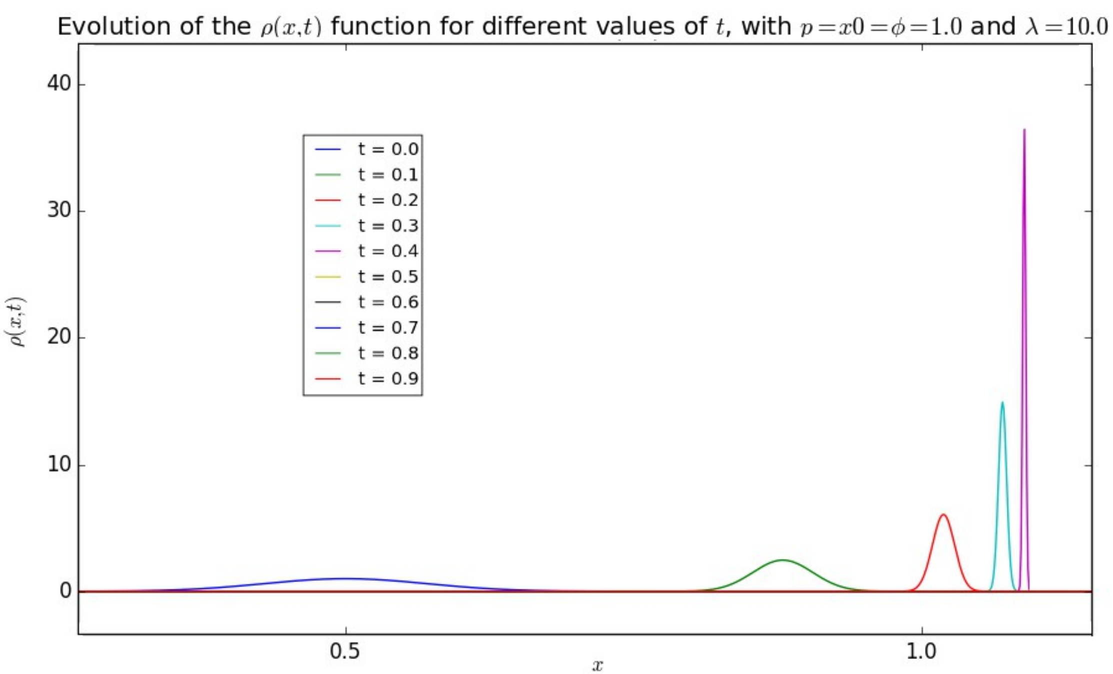
$$\boxed{\frac{\mathrm{d}\rho(x,t)}{\mathrm{d}t} = -\left(\phi + \frac{\mathrm{d}c(x)}{\mathrm{d}x}\right)\rho(x,t) \qquad\qquad \frac{\mathrm{d}x}{\mathrm{d}t} = c(x)}$$

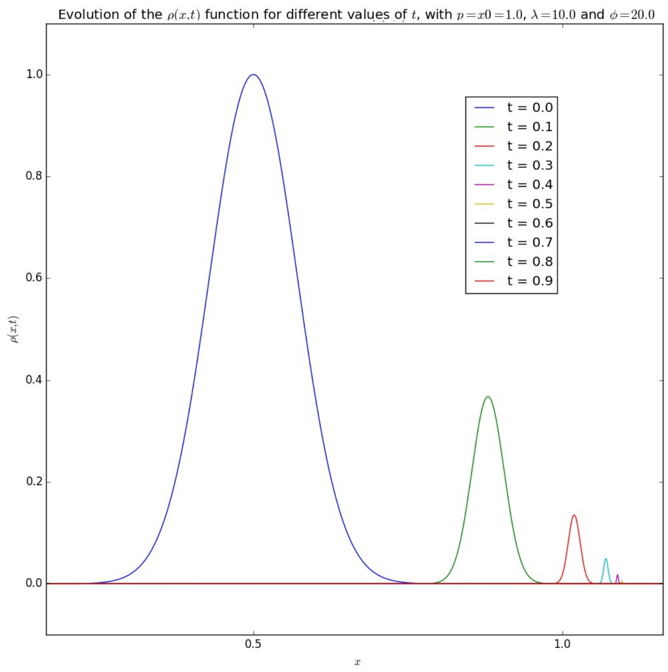
Anaytical Solution

$$x_c(t) = x_0 + \frac{p - K_1 \exp(-\lambda t)}{\lambda}$$

$$\rho_c(t) = K_2 \exp(-(\phi - \lambda)t)$$

$$\rho(x,t) = \rho_0\left(x_0 + \frac{p + [\lambda(x - x_0) - p]\exp(\lambda t)}{\lambda}\right)\exp[-(\phi - \lambda)t]$$





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Computational Solution

$$\frac{\partial u}{\partial t} = R(x,t) - \frac{\partial f(u)}{\partial x}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial R}{\partial t} - \frac{\partial (a(u)R)}{\partial x} + \frac{\partial\left(a(u)\frac{\partial f}{\partial x}\right)}{\partial x}$$

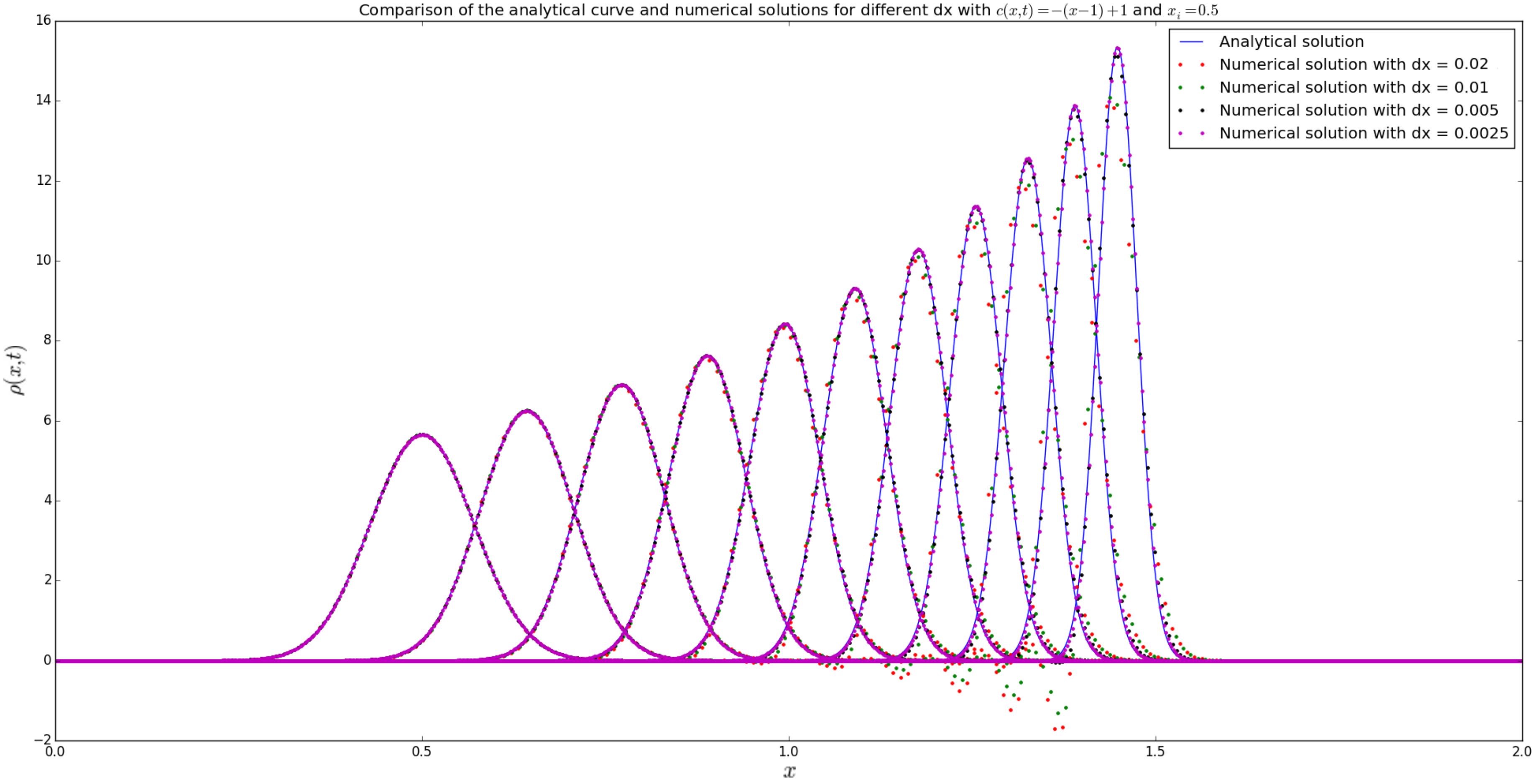
$$u_i^{n+1} \approx u_i^n + \Delta t\left[\frac{\partial u}{\partial t}\right]_i^n + \frac{\Delta t^2}{2}\left[\frac{\partial^2 u}{\partial t^2}\right]_i^n + \ldots$$

$$f(u) = c(x,t)u(x,t)$$

$$R(x,t) = -\phi(x)u(x,t)$$

$$u_i^{n+1} = u_i^n - \frac{s}{2}\left(f_{i+1}^n - f_{i-1}^n\right) + \frac{s}{2}\left[v_{i+1/2}\left(f_{i+1}^n - f_i^n\right) - v_{i-1/2}\left(f_i^n - f_{i-1}^n\right)\right]$$
$$+ \frac{\Delta t}{2}\left[R_i^n + R_i^{n+1}\right] - \frac{\Delta t}{4}\left[v_{i+1/2}\left(R_{i+1}^n - R_i^n\right) - v_{i-1/2}\left(R_i^n - R_{i-1}^n\right)\right]$$

Computational Solution



Computational Solution: Boundary Condition

$$\rho_t(0) = \frac{p_t}{V(0,\rho_t)} = \frac{p_t}{p_t + \lambda\bar{\rho}_t}$$

$$\rho_t(0) \int_0^\infty (\phi(x) + \lambda x)\rho_t(x)\mathrm{d}x = \int_0^\infty \phi(x)\rho_t(x)\mathrm{d}x$$

$$u_0^n \sum_{k=0}^{k_{max}} (\phi(x_k) + \lambda x_k) u_k^n \Delta x = \sum_{k=0}^{k_{max}} \phi(x_k) u_k^n \Delta x$$

$$(u_0^n)^2 \phi(0) + (u_0^n) \left(\sum_{k=1}^{k_{max}} (\phi(x_k) + \lambda x_k) u_k^n - \phi(0) \right) - \sum_{k=1}^{k_{max}} \phi(x_k) u_k^n = 0$$