Modeling neural nets by interacting systems of chains with memory of variable length.

Joint work with A. Duarte, A. Galves, G. Ost

II. Neuromat Workshop, November 2016

Systems of interacting neurons

- Leaky Integrate and fire model with random threshold in high dimension.
- Infinite system of neurons $i \in \mathcal{I}$ that interact.
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• This is what is called **Variable length memory :** the memory of a given neuron goes back in past up to its last spike.

• Spike trains : for each neuron $i \in \mathcal{I}$, we indicate if there is a spike or not at time t, $t \in \mathbb{Z}$.

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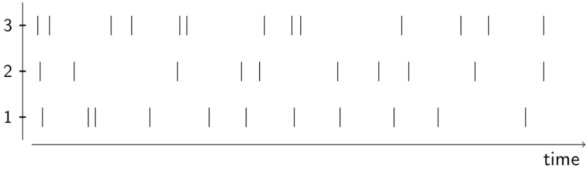

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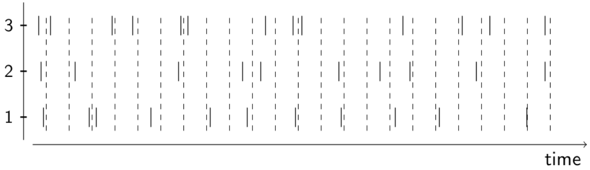

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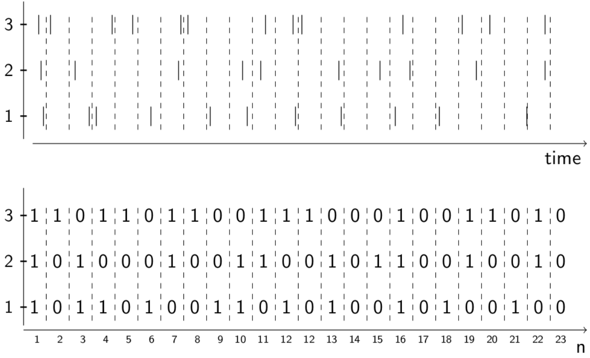

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2. these spikes are weighted by the synaptic weight $W_{j \rightarrow i}$ of neuron j on neuron i
3. they are also weighted by an aging factor which describes the loss of potential since the appearance of the spike of neuron j and the present time t.

The formula for the membrane potential of neuron i at time t :

$$U_t(i) = \sum_j W_{j \to i} \sum_{s=L_t^i+1}^{t-1} g_j(t-s) X_s(j),$$

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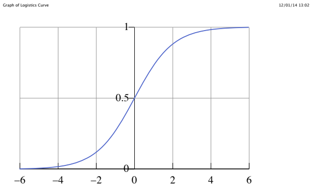
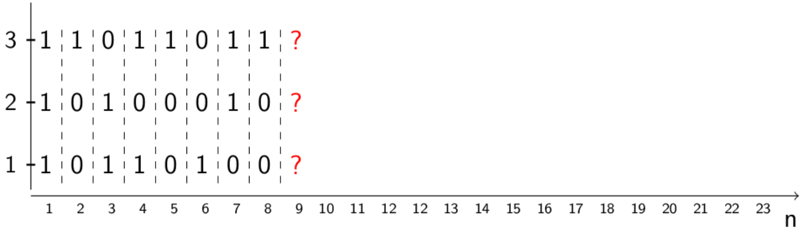
  Thus : $X_{t-1}(i) = 1$ (spike at time $t-1$) $\Rightarrow U_t(i) = 0.$
- $g_j : \mathbb{N} \to \mathbb{R}_+$ describes a leak effect.

$$P(i \text{ spikes at time } t) = \Phi_i(U_t(i) + S_t(i)),$$

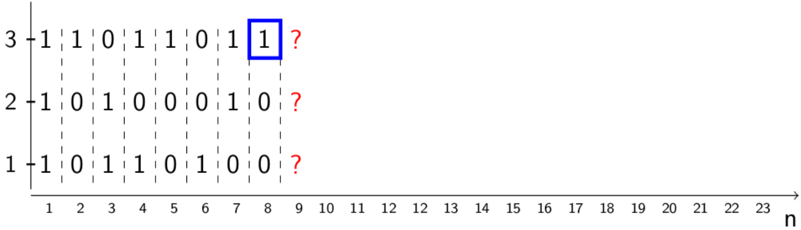
- Φ_i spiking rate function of neuron i : this is an **increasing** function.

It can have a logistic shape.





$$P(X_9(i) = 1|past) = \begin{cases} \phi_i(0), \text{ if } L_9^i = 8 \\ \phi_i\Big(\sum_{j \in I} W_{j \to i} \sum_{s=L_9^i+1}^{8} g_j(t-s)X_s(j)\Big), \text{ otherwise} \end{cases}$$



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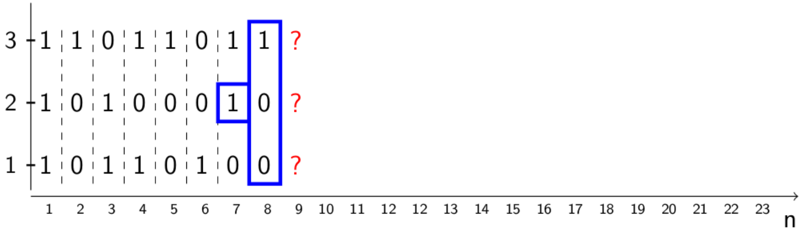

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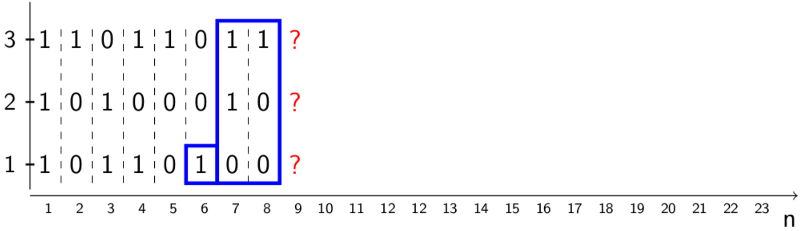

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$$L_9^1 = 6 \Longrightarrow P(X_9(1)=1|past) = \phi_1\big(W_{3\to 1}(g_3(1)+g_3(2)) + W_{2\to 1}g_2(2)\big)$$

Comparison with other models

Our model is a generalization of the classical LIF model, with random thresholds $\Theta_t(i), t \in \mathbb{Z}, i \in I$ which are i.i.d.

$$\phi_i(U_t(i)) = P(U_t(i) > \Theta_t(i))$$

is the probability that the membrane potential $U_t(i)$ exceeds the random threshold $\Theta_t(i)$.

Comparison with other models

If we choose $g_j(1) = 1, g_j(n) = 0$ for all $n \geq 2$ and $\Phi_i(x) = x$, then our model is (almost) the

Kinouchi-Copelli model with only one refractory period :

• each neuron has two states : passive (0) or active (spiking, 1)

• if j has just spiked, then i has a transition from $0 \to 1$ with probability $W_{j\to i}$.

About the model

- Introduced in our joint paper with Antonio Galves in 2013.
- The spiking probability of neuron i depends on the activity of the system since the last spike time.
- The chain $(X_t(i))_{t\in\mathbb{Z}}$ is a **chain with memory of variable length.**
- The model is a **system of interacting chains with memory of variable length.**

More about the model

- Study of the model in **continuous time**, mean-field approximation and **Propagation of chaos** :
  De Masi, Galves, L., Presutti (2015), Fournier and L. (2016), Robert and Touboul (2014), Duarte, Ost and Rodriguez (2016) for a spatially structured model, Drougoul and Veltz (2016), Brochini, Costa, Abadi, Roque, Stolfi and Kinouchi (2016).
- Duarte and Ost (2016) : Finite systems of interacting neurons without external stimulus and with some leak effect stop spiking ...
- Estimation of the spiking rate function : Hodara, Krell, L. (2016)

The interaction graph

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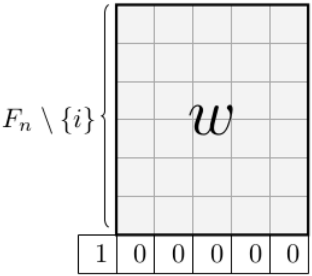
Goal of this talk : Estimate the Interaction neighborhood $\mathcal{V}_i$ of a fixed neuron $i-$ based on an observation of the process in discrete time $[1, \ldots, n]$, within growing spatial windows.

Estimation procedure

AIM : Estimate $\mathcal{V}_i$!

- Growing sequence of finite windows F_n - centered around site i.
- For a test-raster block $w \in \{0,1\}^{\{-\ell,\ldots,-1\}\times F_n\setminus\{i\}}$:

$N_{(i,n)}(w,1)$ counts the number of occurrences of w followed by a spike of neuron i in the sample $X_1(F_n),\ldots,X_n(F_n)$, when the last spike of neuron i has occurred $\ell+1$ time steps before in the past.



	1	2	3	4	5	6	7	8	9	10	11	12	12	13	14	15	16	17	18	19	20	21	22	23
3	1	1	0	1	1	0	1	1	0	0	1	1	1	0	0	0	1	0	0	1	1	0	1	0
2	1	0	1	0	0	0	1	0	0	1	1	0	0	1	0	1	1	0	0	1	0	0	1	0
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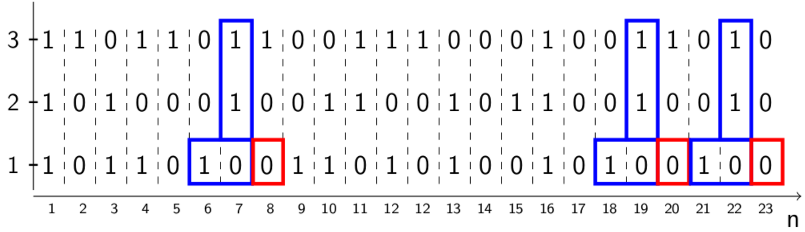
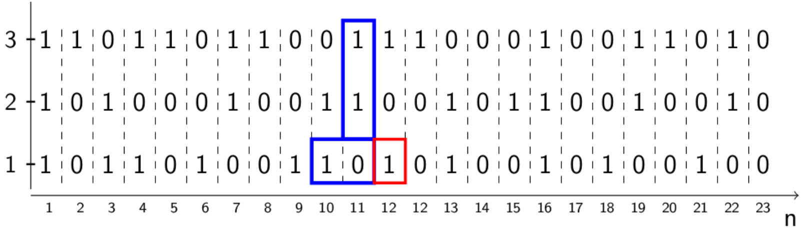
n



$$N_{(1,23)}(w, 0) = \sum_{m=3}^{n} 1\{X_{m-2}^{m-1}(1) = 10, X_{m-1}(F_n \setminus \{1\}) = w, X_m(1) = 0\}.$$



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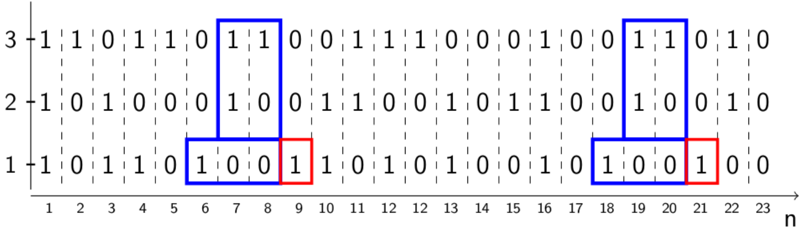

$$N_{(1,23)}(w, 1) = \sum_{m=4}^{n} 1\{X_{m-3}^{m-1}(1) = 10^2, X_{m-2}^{m-1}(F_n \setminus \{1\}) = w, X_m(1) = 1\}.$$

- Estimated spiking probability $\hat{p}_{(i,n)}(1|w) = \frac{N_{(i,n)}(w,1)}{N_{(i,n)}(w)}$.

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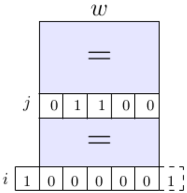
- Test statistics to test the influence of neuron j on neuron i :

$$\Delta_{(i,n)}(j) = \max_{w,v:v_{\{j\}^c}=w_{\{j\}^c}} |\hat{p}_{(i,n)}(1|w) - \hat{p}_{(i,n)}(1|v)|.$$



Definition

For any positive threshold parameter $\epsilon > 0$, the estimated interaction neighborhood of neuron $i \in F_n$, at accuracy ϵ, given the sample $X_1(F_n), \ldots, X_n(F_n)$, is defined as

$$\hat{V}^{(\epsilon)}_{(i,n)} = \{j \in F_n \setminus \{i\} : \Delta_{(i,n)}(j) > \epsilon\}.$$

Conditions

1) Spiking rate functions ϕ_i *are strictly increasing, Lipschitz*

$$|\phi_i(z) - \phi_i(z')| \leq \gamma |z - z'|$$

and bounded from above and below. 2) Uniform summability of the synaptic weights

$$r := \sup_i \sum_j |W_{j \to i}| < \infty.$$

3) $\sup_i g_i(n) < \infty$ *for all* n.

Theorem

Fix $i \in \mathcal{I}$ and suppose :

- *V_i finite and*
- *$|F_n| = o(\log n)$.*

Let $X_1(F_n), \ldots, X_n(F_n)$ be a sample produced by the stochastic chain $(X_t)_{t \in \mathbb{Z}}$ satisfying our assumptions. Then for $\epsilon_n = O(n^{-\xi/2})$, for some $\xi > 0$,

$$\hat{V}_{(i,n)}^{(\epsilon_n)} = V_i \text{ eventually almost surely.}$$

Can be extended to the case when the interaction neighborhood of i is infinite !

Remarks

- We are not estimating the weights of interaction, only the existence or not of interactions. (An oriented graph which is not weighted !)
- We are able to estimate the existence of interactions without assuming the knowledge of the spiking probability functions or the leak functions.
- We can deal with the case in which the system has several invariant states.

Open questions

• Multi-unit recordings of spiking activity show only a picture of a very tiny part of the brain ...

• What does the observation of a very small part of the net tells us about the global behavior of the system ?

• In this talk : estimation of the anatomical graph of interactions between single neurons.

• What about the graph describing the functional interactions betweens components or regions of the brain ?

– Usually derived from correlations

– absence of correlations is not equivalent with independence ! ! ! ! (this would only be true for Gaussian systems)

– need : new mathematical definition of functional dependance.

• Take into account the **characteristics of the graph of interactions (anatomic and/or functional) of the brain.**

• Types of graphs that have been considered : critical Erdös-Rényi random graphs, small world, rich-club networks, ...

• Statistical model selection for systems of spiking neurons with interaction graphs belonging to one of these classes ?

Study of the relaxation period

• Relaxation period = interval during which the system transits from an initial condition to an asymptotic state (= stable state).

• Study how the limit law of the last spiking time (which is finite, see Duarte and Ost (2016)), rescaled by its mean value, depends on the leak rate for a fixed spiking rate function.

• Is there a notion of **criticality** as in Kinouchi and Copelli (Optimal Dynamical Range of Excitable Networks at Criticality, 2006) - and an associated dynamical phase transition ?

Some literature

Paper is on arXiv :

• Duarte, A., Galves, A., L.E., Ost, G., Estimating the interaction graph of stochastic neural dynamics. 2016, arXiv.

Thank you for your attention.