Random graphs (a droplet)

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NeuroMat—First Workshop

IME/USP

20 January 2014

Partially supported by CNPq (477203/2012-4, 308509/2007-2) and
by FAPESP (2013/07699-0, 2013/03447-6)

Aim of talk

A glimpse of the theory of random graphs

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- The Erdős–Rényi random graph

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A glimpse of the theory of random graphs

- The Erdős–Rényi random graph
- A directed variant

Outline of the talk

- ▷ Preliminaries

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  - Graphs

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- The Erdős–Rényi random graph
- The phase transition
- A version for directed graphs

Graphs

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  - ◦ V: set of *vertices*
  - ◦ E: set of *edges* (= unordered pairs of vertices)

A graph

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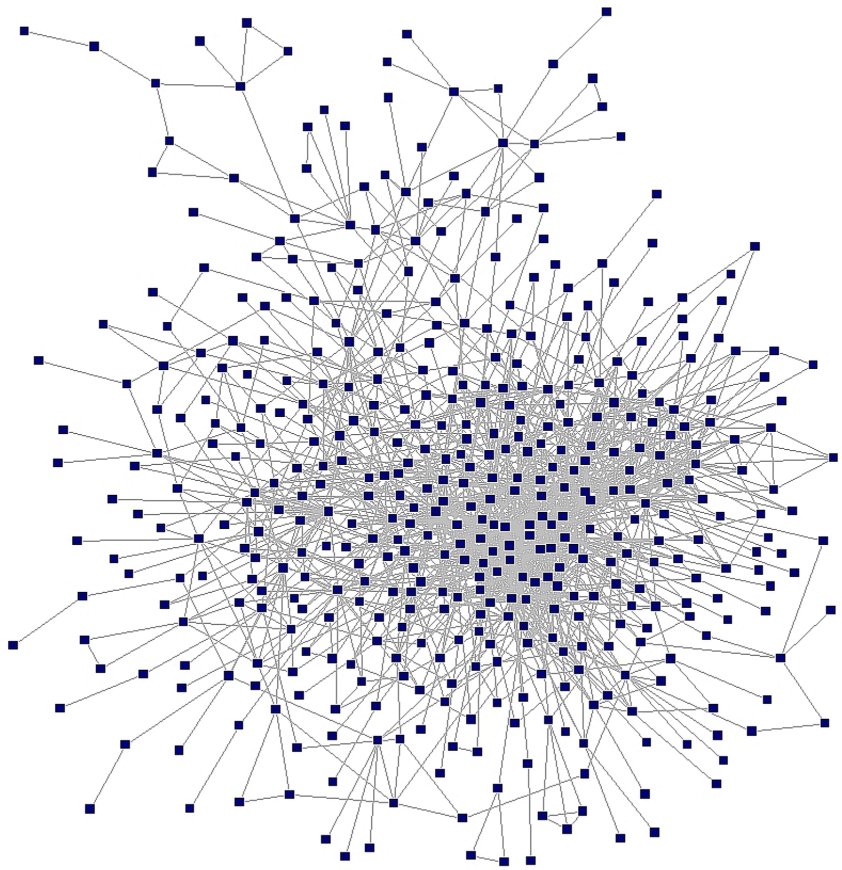

By V. Krebs, from `http://www.orgnet.com/Erdos.html`

Random graphs

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- Uniform model on $\binom{\binom{[n]}{2}}{m}$
- $G(n, p)$: binomial variant; $0 \leq p = p(n) \leq 1$

Phase transition: $G(n, p)$

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Theorem 1 (Łuczak (1990), building on Bollobás (1984))**.** *Let* $np = 1 + \varepsilon$*, where* $\varepsilon = \varepsilon(n) \to 0$ *but* $n|\varepsilon|^3 \to \infty$*, and* $k_0 = 2\varepsilon^{-2} \log n|\varepsilon|^3$*.*

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(i) If $n\varepsilon^3 \to -\infty$, *then* $G(n, p)$ *a.a.s. contains no component of order greater than* k_0. *Moreover, a.a.s. each component of* $G(n, p)$ *is either a tree, or contains precisely one cycle.*

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(i) If $n\varepsilon^3 \to -\infty$*, then* $G(n, p)$ *a.a.s. contains no component of order greater than* k_0*. Moreover, a.a.s. each component of* $G(n, p)$ *is either a tree, or contains precisely one cycle.*

(ii) If $n\varepsilon^3 \to \infty$*, then* $G(n, p)$ *a.a.s. contains exactly one component of order greater than* k_0*. This component a.a.s. has* $(2+o(1))\varepsilon n$ *vertices.*

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▷ Binomial directed graph: $D(n, p)$

Phase transition: $D(n, p)$

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(i) *If* $\varepsilon^3 n \to -\infty$, *then a.a.s. every strong component in* $D(n,p)$ *is either a vertex or a cycle of length* $O(1/|\varepsilon|)$.

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(i) If $\varepsilon^3 n \to -\infty$*, then a.a.s. every strong component in* $D(n,p)$ *is either a vertex or a cycle of length* $O(1/|\varepsilon|)$*.*

(ii) If $\varepsilon^3 n \to \infty$*, then a.a.s.* $D(n,p)$ *contains a unique complex component, of order* $(4+o(1))\varepsilon^2 n$*, whereas every other strong component is either a vertex or a cycle of length* $O(1/\varepsilon)$*.*