# Estimating the spiking rate for a model of interacting neurons.

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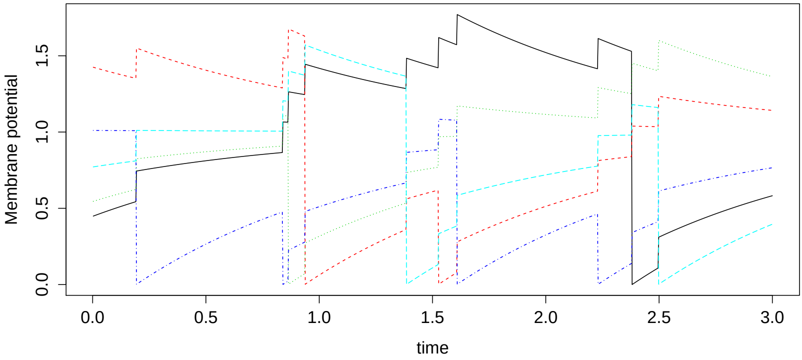
Second NeuroMat Workshop: New frontiers in neuromathematics

# A Piecewise deterministic Markov Process

**fig. 1**



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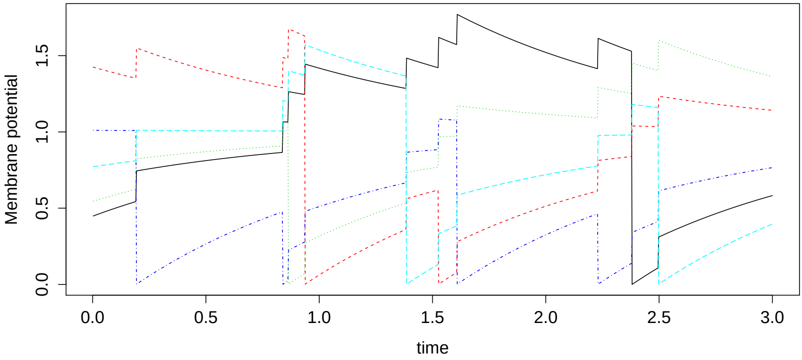
$$L\varphi(x) = \sum_{i=1}^N f(x_i) \left[\varphi(\Delta_i(x)) - \varphi(x)\right] - \lambda \sum_i \left( \frac{\partial \varphi}{\partial x_i}(x) \left[x_i - m\right] \right),$$

with

$$(\Delta_i(x))_j = \left\{ \begin{array}{ll} x_j + \frac{1}{N} & j \neq i \\ 0 & j = i \end{array} \right\}.$$

The aim is to build an estimator for the intensity function $f$ from the observation of the trajectory up to a time $t$.

fig. 1



# definition of the estimator

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Jump measure : $\mu(dt, dx) = \sum_{i=1}^{N} \sum_{n \geq 1} 1_{\{T_n^i < \infty\}} \delta_{(T_n^i, X_{T_n^i -}^i)}(dt, dx).$

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Occupation time measure : $\eta(A \times B) = \int_A \left( \sum_{i=1}^{N} 1_B(X_s^i) \right) ds$

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in the following way :

(2)
$$\hat{f}_{t,h}(a) = \frac{\int_0^t \int_{\mathbb{R}} Q_h(y-a)\mu(ds,dy)}{\int_0^t \int_{\mathbb{R}} Q_h(y-a)\eta(ds,dy)}, \text{ avec } Q_h(y) := \frac{1}{h} Q\left(\frac{y}{h}\right).$$

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We are interested in the error in $L^2$ uniformly on a given Hölder class of functions $H(\beta, F, L, f_{min})$ of order $\beta = k + \alpha$.

# Ergodicity

**Theorem**

*The process X is positif Harris reccurent, with unique invariant measure $\pi$, i.e. for all $B \in \mathcal{B}([0,K]^N)$,*

$$\pi(B) > 0 \ \textit{ implies } P_x\left(\int_0^\infty 1_B(X_s)ds = \infty\right) = 1$$

*for all $x \in [0,K]^N$.*
*Moreover, there exists constants $C > 0$ and $\kappa > 1$ such that*

$$\sup_{f \in H(\beta,F,L,f_{min})} \|P_t(x,\cdot) - \pi\|_{TV} \leq C\kappa^{-t}.$$

# Definitions

Recall the definition of the estimator :

$$\hat{f}_{t,h}(a) = \frac{\int_0^t \int_{\mathbb{R}} Q_h(y-a)\mu(ds,dy)}{\int_0^t \int_{\mathbb{R}} Q_h(y-a)\eta(ds,dy)}.$$

This estimator is well defined on events of the type

$$A_{t,r} := \left\{ \frac{1}{Nt} \int_0^t \int_{\mathbb{R}} Q_h(y-a)\eta(ds,dy) \geq r \right\}.$$

# Convergence of the estimator

**Theorem**

*There exists a constant $r^* > 0$ such that,*
*(i) for $h_t := t^{-\frac{1}{2\beta+1}}$, for all $x \in [0, K]^N$,*

$$\lim\sup_{t\to\infty} \sup_{f\in H(\beta,F,L,f_{min})} t^{\frac{2\beta}{2\beta+1}} E_x^f \left[ |\hat{f}_{t,h_t}(a) - f(a)|^2 | A_{t,r^*} \right] < \infty.$$

*(ii) Moreover, for $h_t = o(t^{-1/(1+2\beta)})$, and for all $f \in H(\beta, F, L, f_{min})$ we have the weak convergence under $P_x^f$ :*

$$\sqrt{th_t} \left( \hat{f}_{t,h_t}(a) - f(a) \right) \to \mathcal{N}(0, \Sigma(a))$$

*with $\Sigma(a) = \frac{f(a)}{N\pi_1(a)} \int Q^2(y)dy$.*

# Optimal speed of convergence

**Theorem**

*For all* $a \in S_{d,k}$ *and* $x \in [0, K]^N$, *we have*

$$\liminf_{t\to\infty} \inf_{\hat{f}_t} \sup_{f \in H(\beta, F, L, f_{min})} t^{\frac{2\beta}{1+2\beta}} E_x^f[|\hat{f}_t(a) - f(a)|^2] > 0,$$

*where inf is considered on the set of all possible estimators* $\hat{f}_t(a)$ *for* $f(a)$.

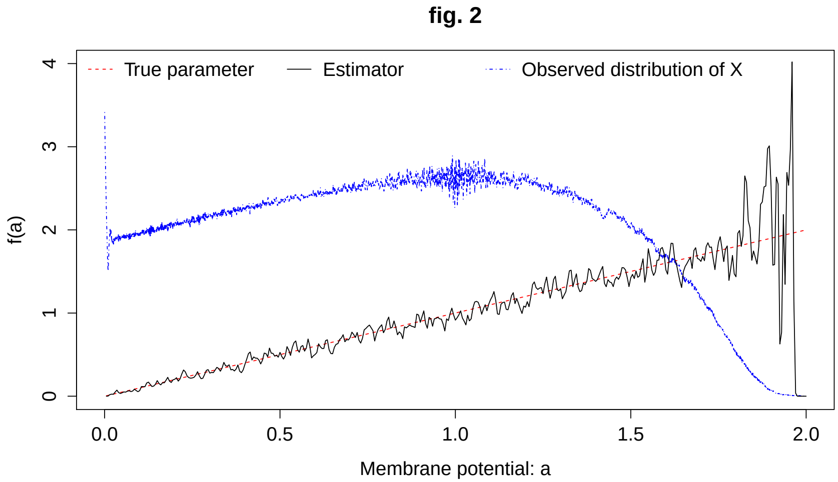
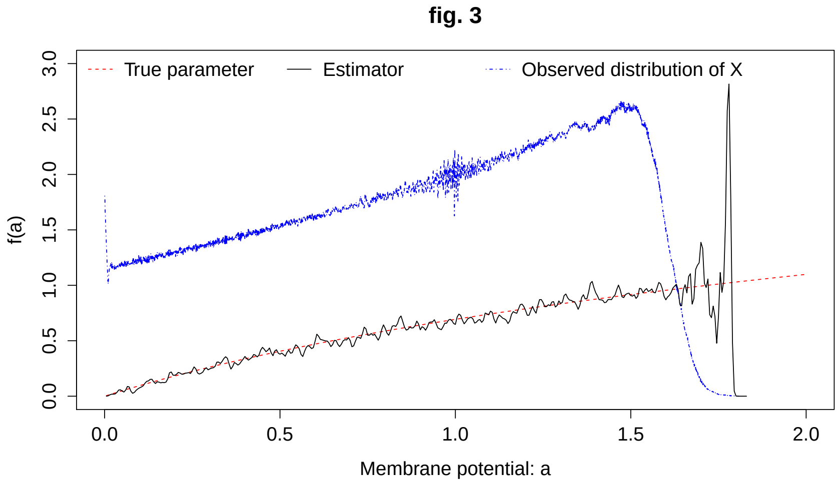


fig. 2
True parameter
Estimator
Observed distribution of X
f(a)
0
1
2
3
4
0.0
0.5
1.0
1.5
2.0
Membrane potential: a

# Practical results on simulations



# Practical results on simulations