On the CLT in planar oriented percolation
NeuroMat Workshop

Achillefs Tzioufas
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A. Broadbent and Hammersley (57). Let $\mathscr{G}(\mathscr{L}, \mathbb{B})$ *directed graph* with vertices

$$\mathscr{L} = \{(x, n) \in \mathbb{Z}^2 : x + n \in 2\mathbb{Z} \text{ and } n \geq 0\},$$

$2\mathbb{Z} = \{2k : k \in \mathbb{Z}\}$, and edges $\mathbb{B} = \{[(x, n), (y, n+1)\rangle : |x - y| = 1\}$.

Retaining parameter $p \in (0, 1)$. Let $\omega = (\omega(b) : b \in \mathbb{B})$ be i.i.d. $\{0, 1\}$ r.v.'s: 1 with pr. p, and 0 otherwise.

$$\xi_n^\eta(\omega) = \{x : (y, 0) \to (x, n), \text{ for some } y \in \eta\}, \ \eta \subseteq 2\mathbb{Z}. \tag{1}$$

$$\xi_n^\eta(x) = 1(0) \text{ if } x \in \xi_n^\eta (x \notin \xi_n^\eta),$$

where $x \in 2\mathbb{Z}(n \text{ even})$ and $x \in 2\mathbb{Z} + 1(n \text{ odd})$.

B. Let O be the *origin*. The *Percolation event*

$$\Omega_\infty := \cap_{n\geq 1}\Omega_n = \{|\xi_n^O| \geq 1, \text{ for all } n \geq 1\}, \ \Omega_n := \{|\xi_n^O| \geq 1\}, \tag{2}$$

Let in addition $\rho(p)$ be the *asymptotic density.*

$$\rho(p) = \mathbb{P}(\Omega_\infty) = \lim_{n\to\infty} \mathbb{P}(\Omega_n) = .. = \lim_{n\to\infty} \mathbb{P}(\xi_n^{2\mathbb{Z}} \cap \{O\} \neq \emptyset) \tag{3}$$

The *critical value*

$$p_c = \inf\{p : \rho(p) > 0\}, \tag{4}$$

Harris (78) shows that, if $p > \frac{8}{9}$, then

$$\inf_{n>0} \frac{|\xi_n^O|}{n} > 0 \text{ a.s. on } \Omega_\infty.$$

C. Durrett (80). Let $r_n = \sup \xi_n^O$ and $l_n = \inf \xi_n^O$. If $p > p_c$ there is an *asymptotic velocity* $\alpha = \alpha(p) > 0$

$$\lim_{n\to\infty} \frac{r_n}{n} = \lim_{n\to\infty} \frac{l_n}{n} = \alpha \text{ a.s. on } \Omega_\infty.$$

Clearly

$$|\xi_n^O| = \sum_{x=l_n}^{r_n} \xi_n^O(x).$$

Durrett and Griffeath (83). If $p > p_c$, then

$$\lim_{n\to\infty} \frac{\sum_{x=l_n}^{r_n} \xi_n^O(x)}{n} = \alpha\rho \text{ a.s. on } \Omega_\infty \quad (*) \text{ [LLN]}$$

*Intuition Durrett (80). If all processes are defined on the same probability space

$$|\xi_n^O| = \sum_{x=\bar{l}_n}^{\bar{r}_n} 1(x \in \xi_n^{2\mathbb{Z}}), \text{ a.s. on } \Omega_\infty \tag{5}$$

D. Let s_n be the *span* of $\xi_n^O \cap \mathscr{L}$, so $|s_n| = \frac{r_n - l_n}{2} + 1$, and $\lim_{n\to\infty} \frac{|s_n|}{n} = \alpha$ a.s. on Ω_∞.

Theorem 0.1 (T. (16).)**.** *Let $p > p_c$.*[1] *We have that*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\sum_{x\in s_n} \xi_n^O(x) - |s_n|\rho_n}{\sigma\sqrt{|s_n|}} \le x \,\middle|\, \Omega_\infty \right) \longrightarrow \int_{-\infty}^{x} (2\pi)^{-1/2} e^{-u^2/2} du \;\; (**) \;\; \textit{[CLT]} \tag{6}$$

as $n \to \infty$, where $\sigma^2 = \sum_x \mathrm{Cov}(x \in \xi^{\bar{\nu}}, O \in \xi^{\bar{\nu}}) < \infty$, and $\xi_{2n}^* \Rightarrow \xi^{\bar{\nu}}$, as $n \to \infty$.

Ergodic improvement of $(*)$,

$$\lim_{n\to\infty} \frac{\sum_{x=l_n}^{r_n} f(\xi_n^O(x))}{n} = \alpha \mathbb{E} f(\xi^{\bar{\nu}}(0)) \text{ a.s. on } \Omega_\infty, \tag{7}$$

for any $f : \{0,1\} \to [0,\infty)$ such that $\mathbb{E} f(\xi^{\bar{\nu}}(0)) < \infty$.

E. ** *Heuristics* [asymptotic independence]

$$\begin{aligned} \frac{1}{\sqrt{n}} \left(|\xi_n^{2\mathbb{Z}} \cap [\bar{l}_n, \bar{r}_n]|| - |\xi_n^{2\mathbb{Z}} \cap [-\alpha n, \alpha n]|| \right) \;\; &\approx \;\; \frac{1}{\sqrt{n}} \left(\rho_n \frac{\bar{r}_n - \alpha n}{2} + \rho_n \frac{+\alpha n - \bar{l}_n}{2} \right) \\ &\Rightarrow \;\; 4\rho N(0, \sigma^2_{edges}) \end{aligned}$$

Turns out that *Counter-intuitively* (extending Anscombe (52))

$$\frac{1}{\sqrt{n}} |\xi_n^{2\mathbb{Z}} \cap [\bar{l}_n, \bar{r}_n]|| - |\xi_n^{2\mathbb{Z}} \cap [-\alpha n, \alpha n]|| \xrightarrow{p} 0.$$

[1] Note that, thanks to the work by Bezuidenhoot and Grimmett (90), we know that $\rho(p_c) = 0$, and therefore, the assumption $p > p_c$ may be replaced by $\rho(p) > 0$.