Bootstrap percolation

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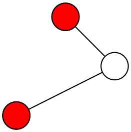

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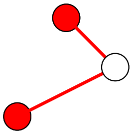

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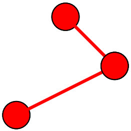

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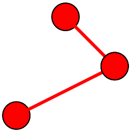

Link

Small exercise:
Consider the 2-neighbour bootstrap percolation on $n \times n$ grid.

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What is the minimal number of sites that can lead to percolation?

▸ Link

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Sharp metastability threshold for two-dimensional bootstrap percolation, *Probab. Theory Related Fields*, 125 (2003), pp. 195-224.

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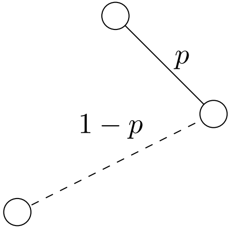

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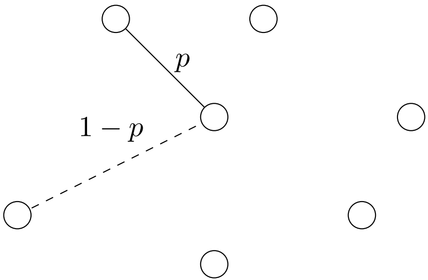

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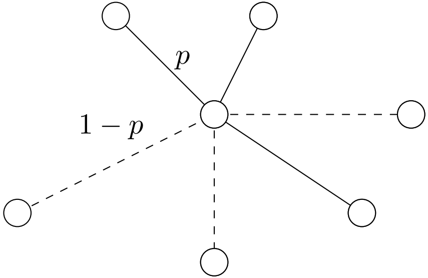

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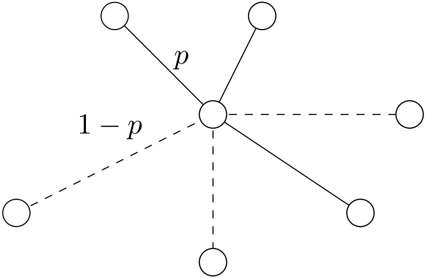

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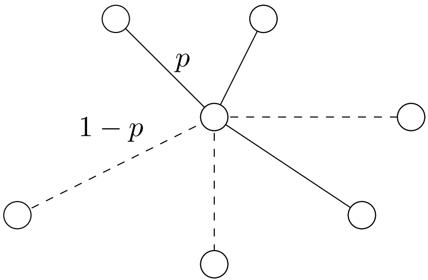

$$\deg(v) \sim \text{Bin}(n-1,p) \qquad \mathbb{E}\left(\deg(v)\right) = (n-1)p$$

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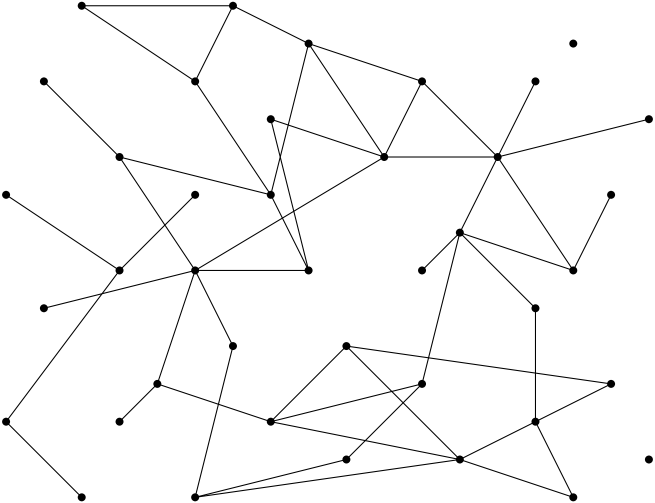

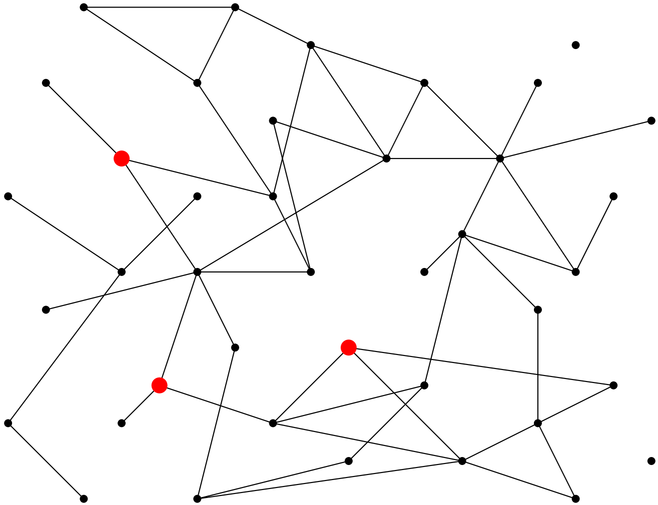

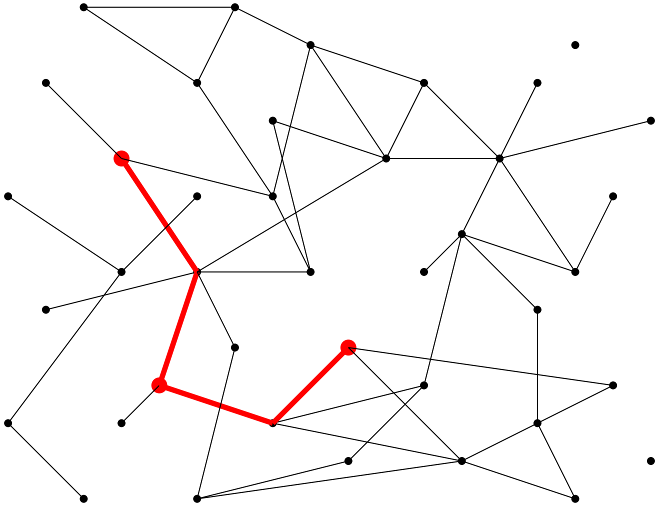

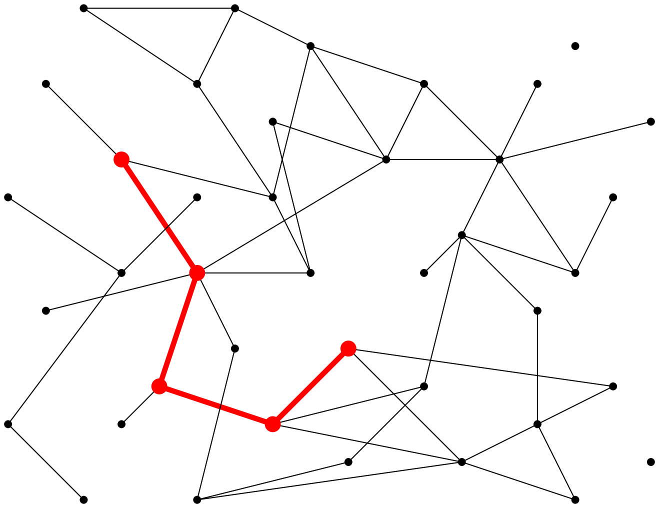

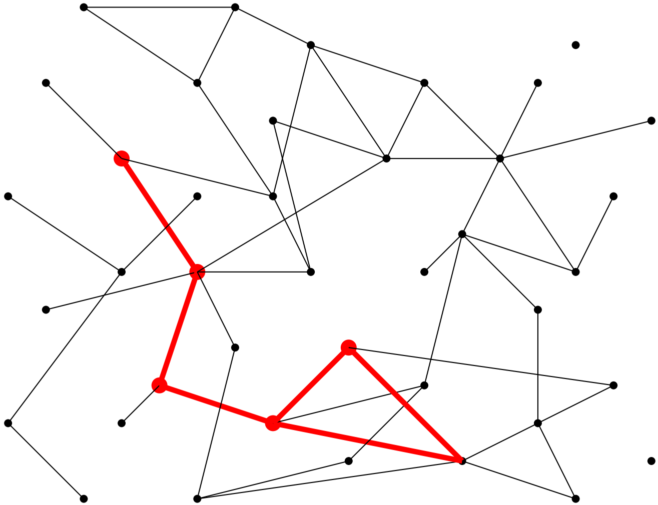

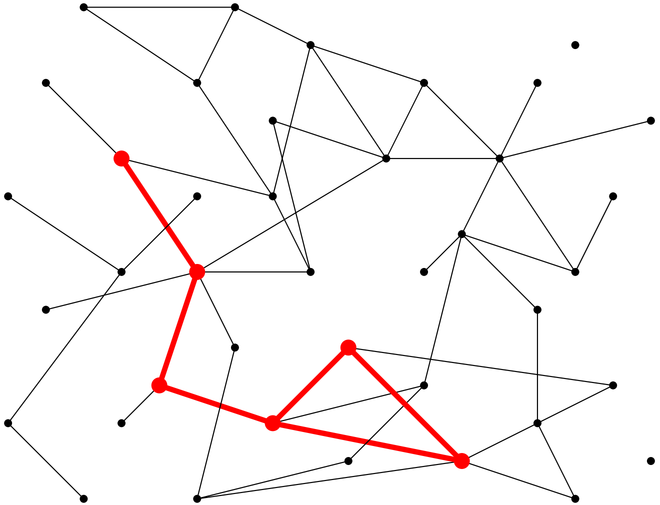

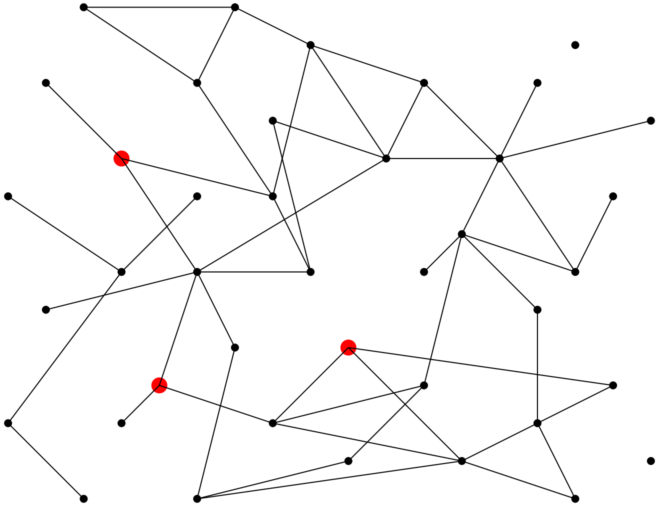

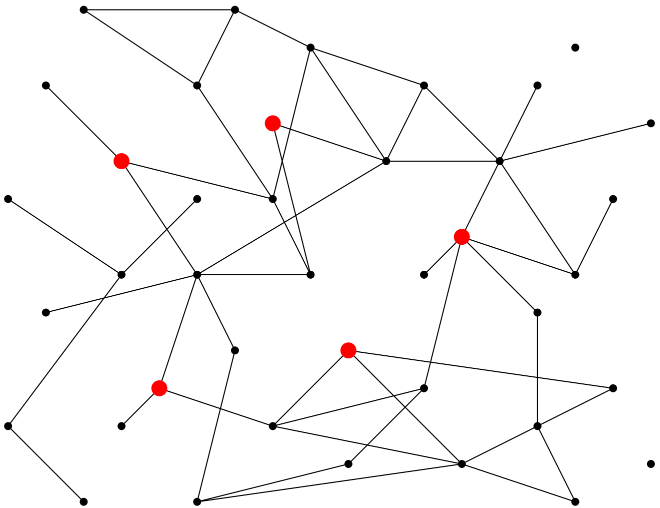

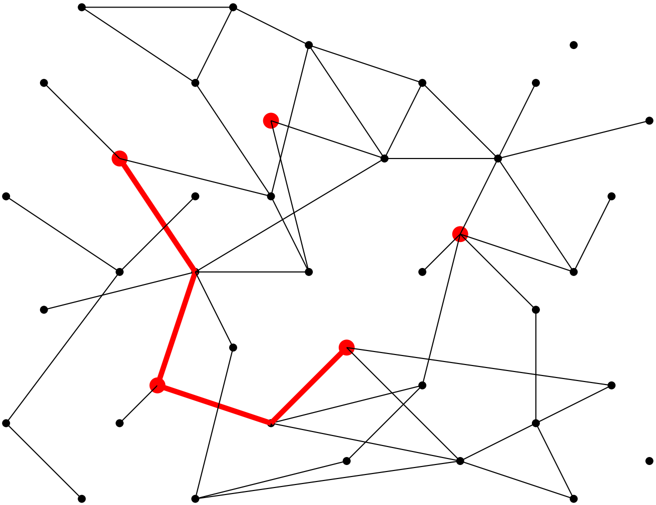

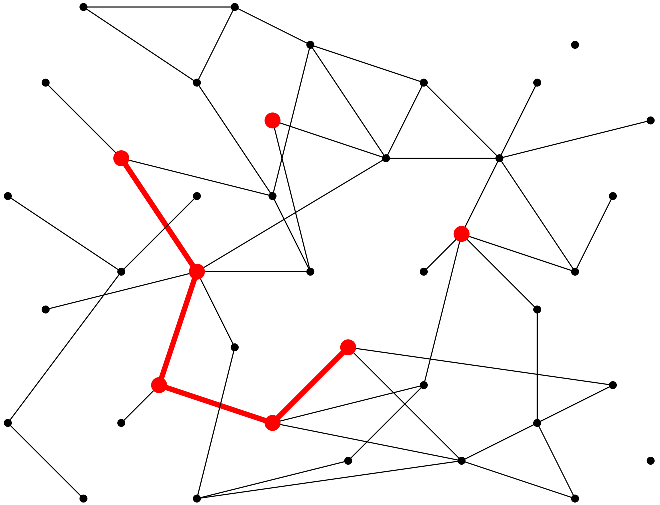

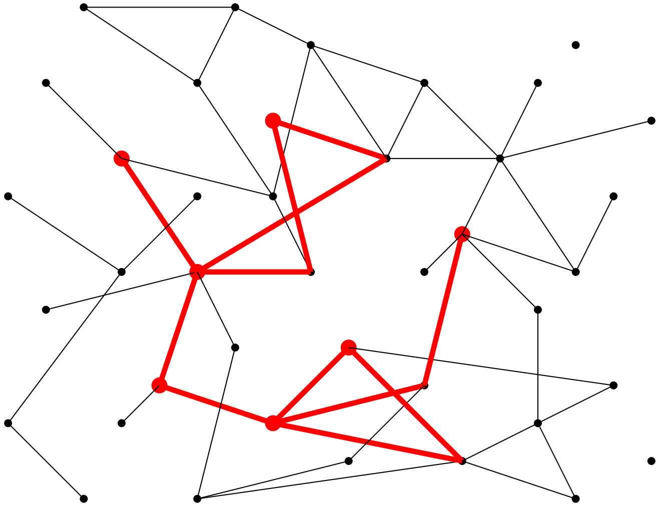

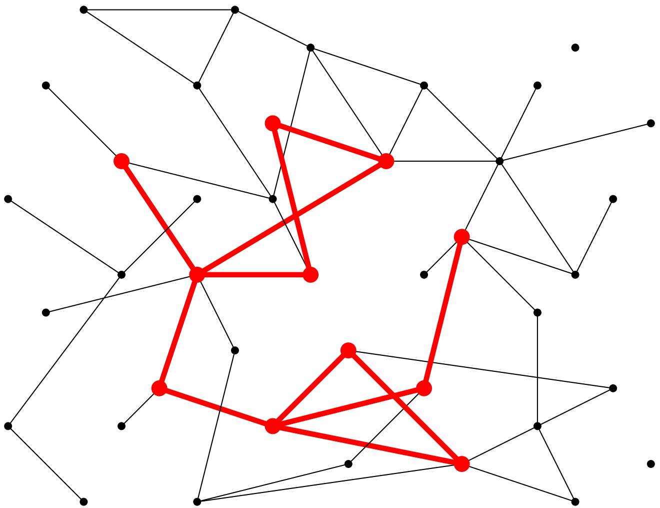

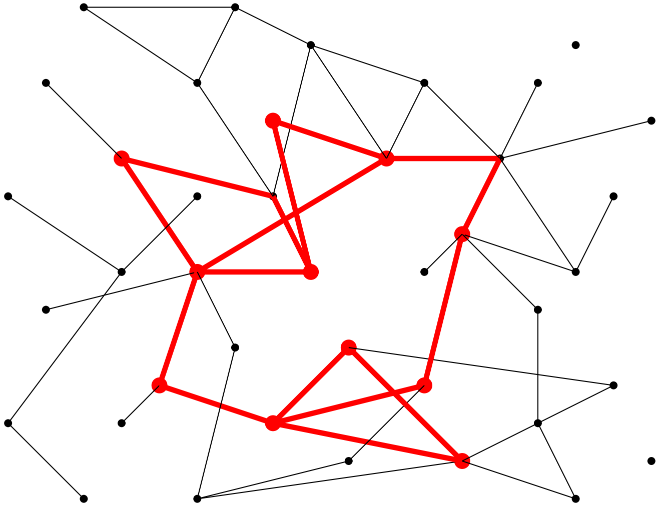

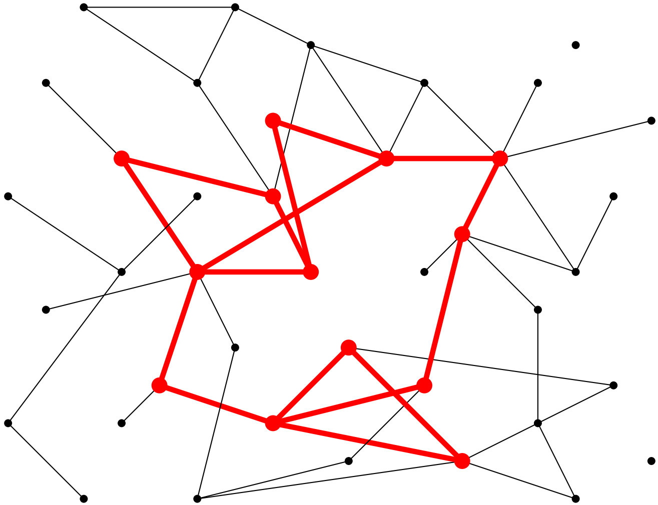

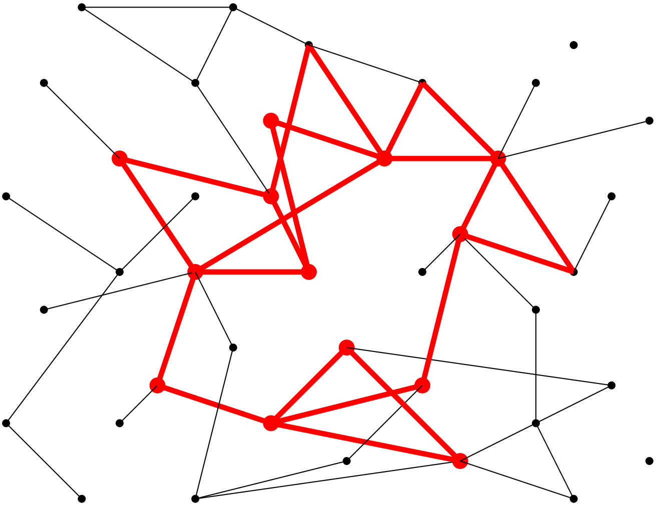

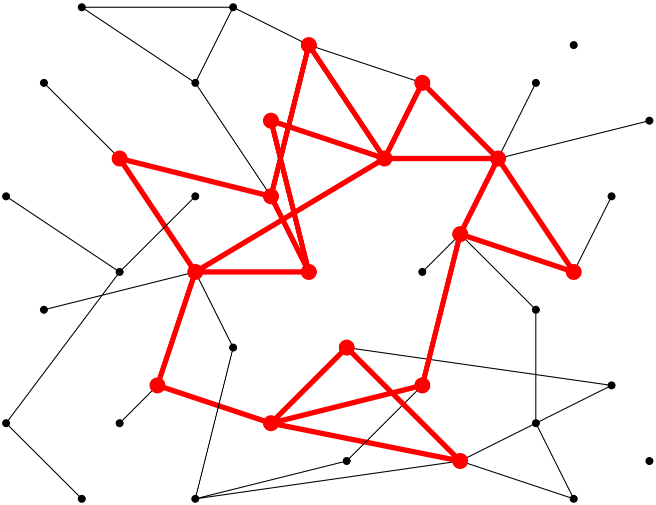

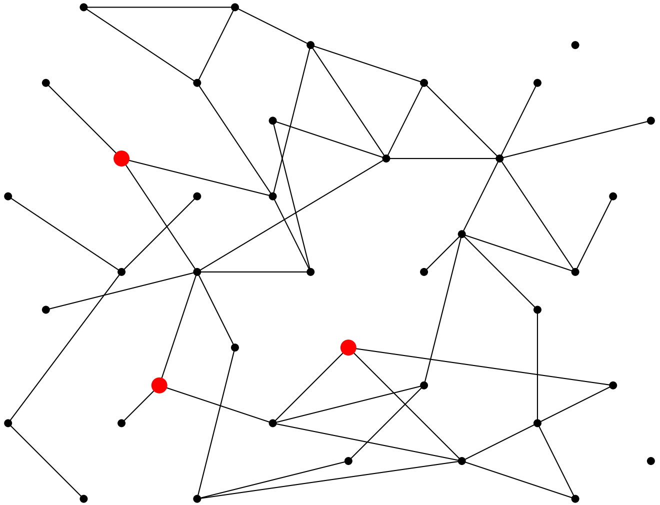

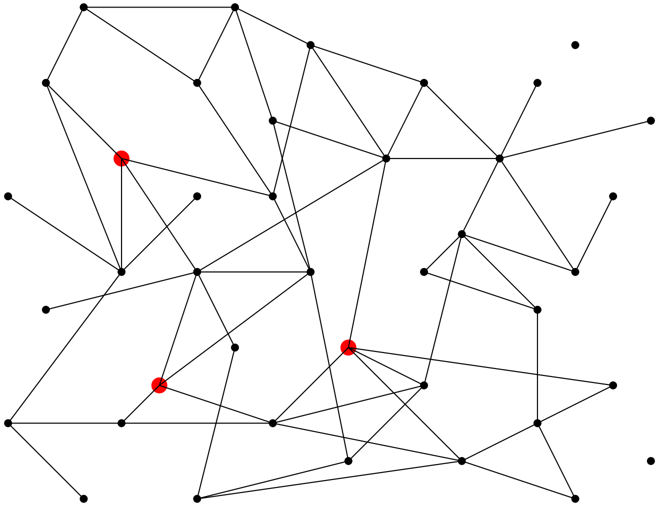

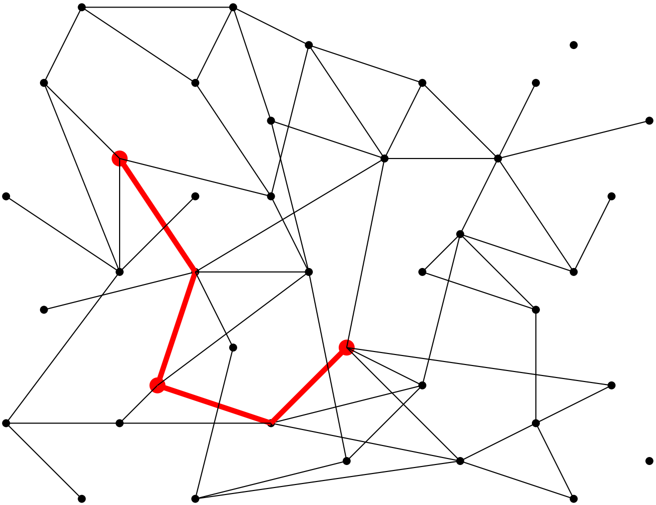

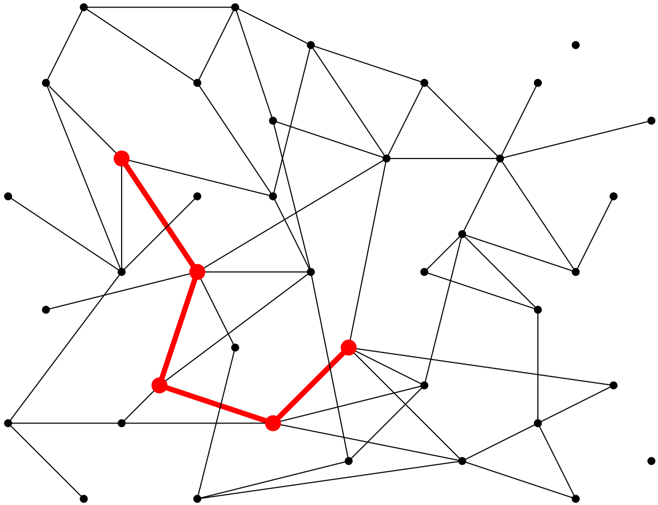

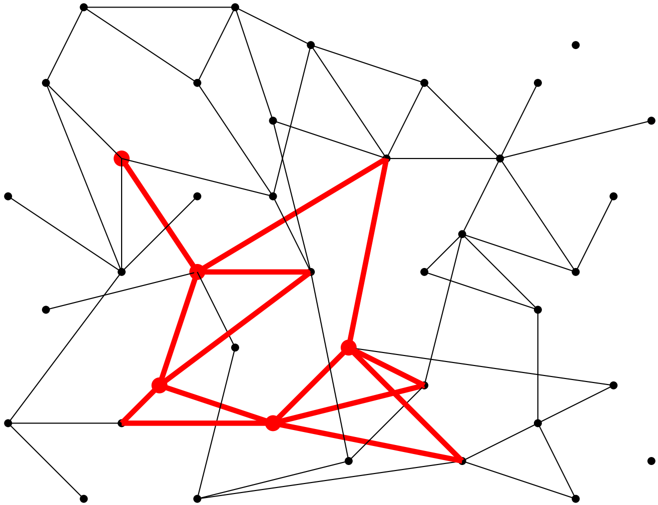

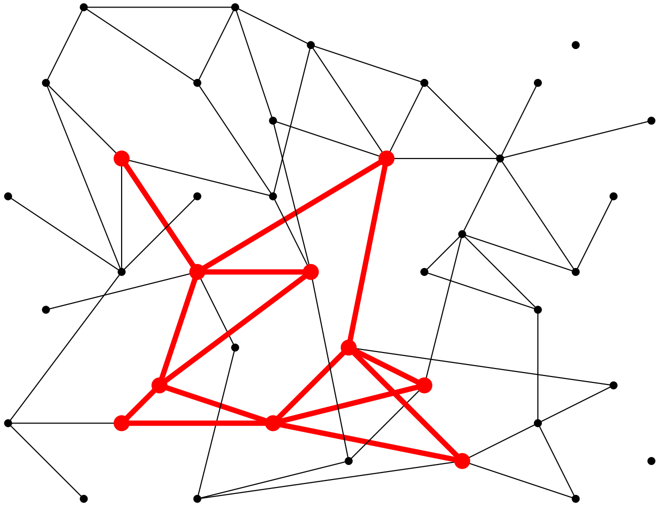

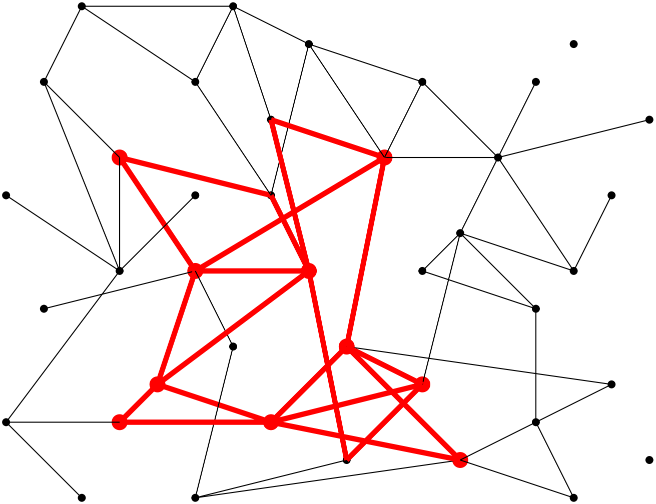

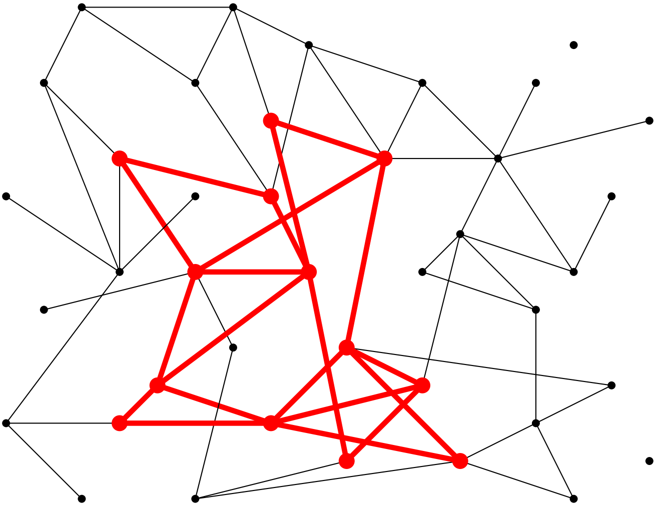

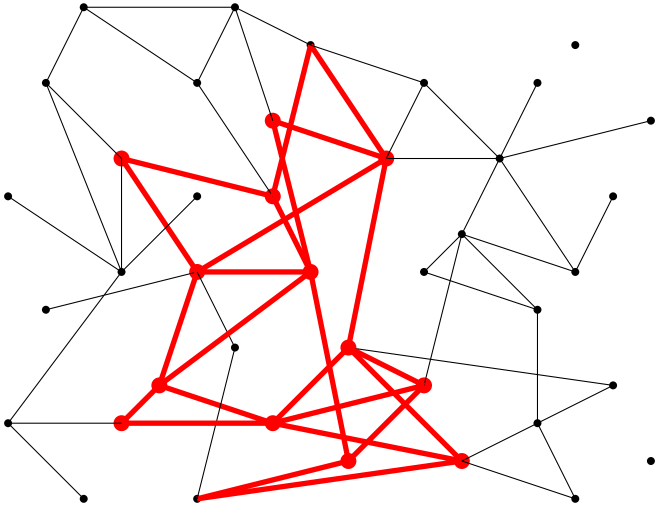

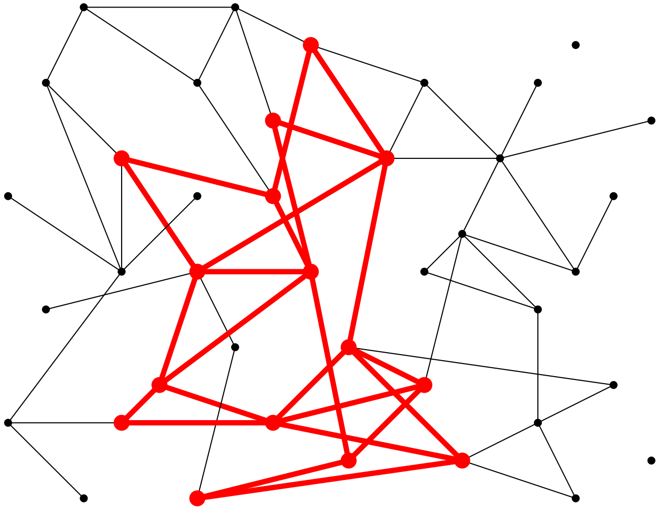

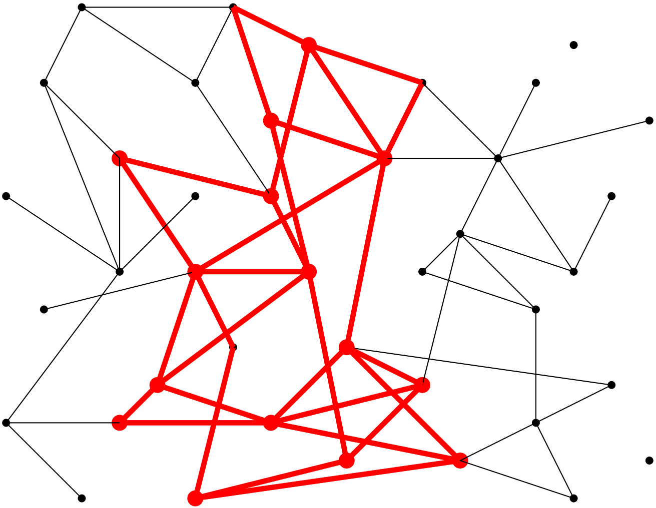

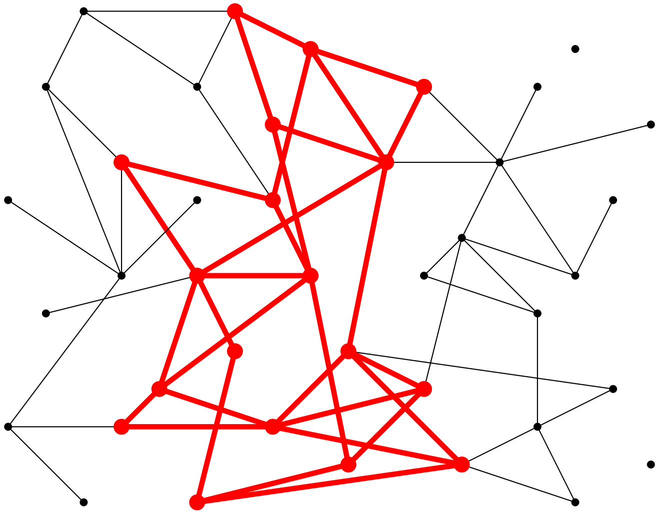

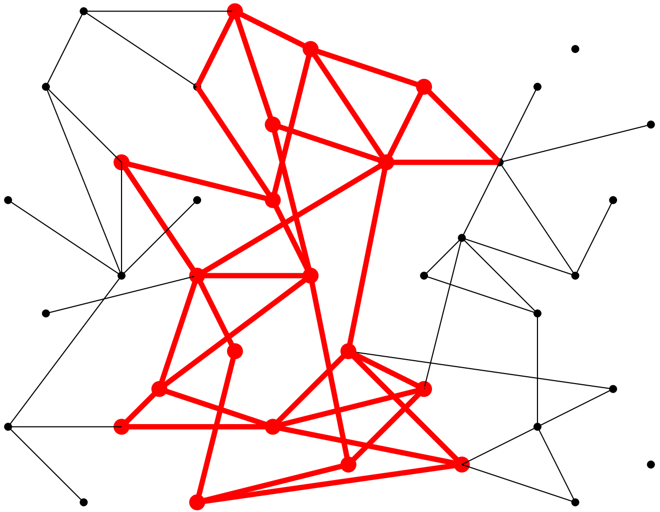

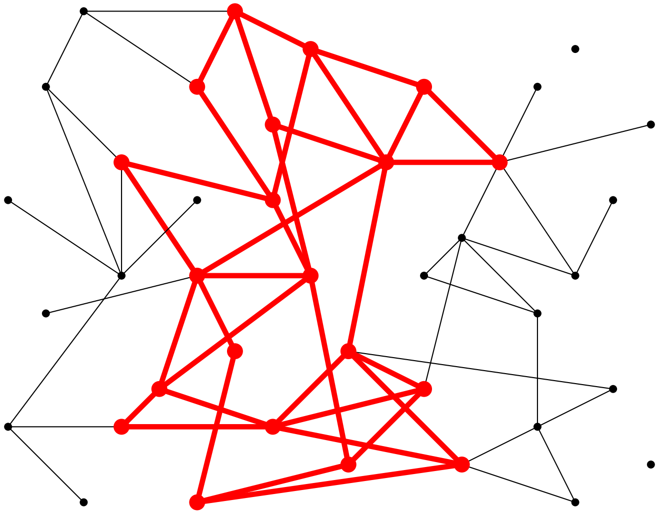

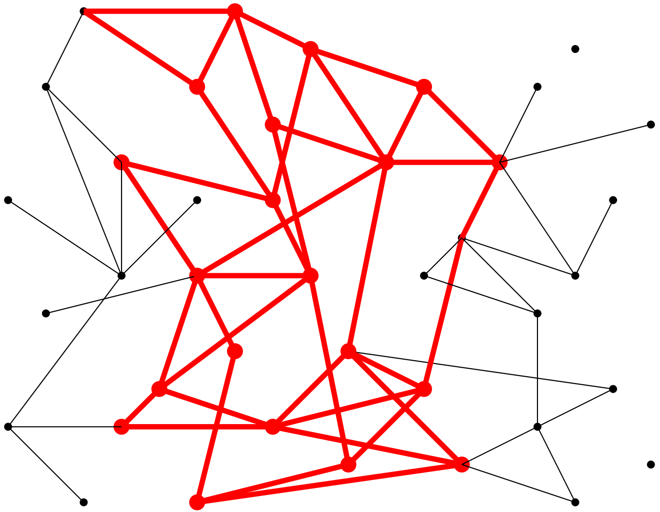

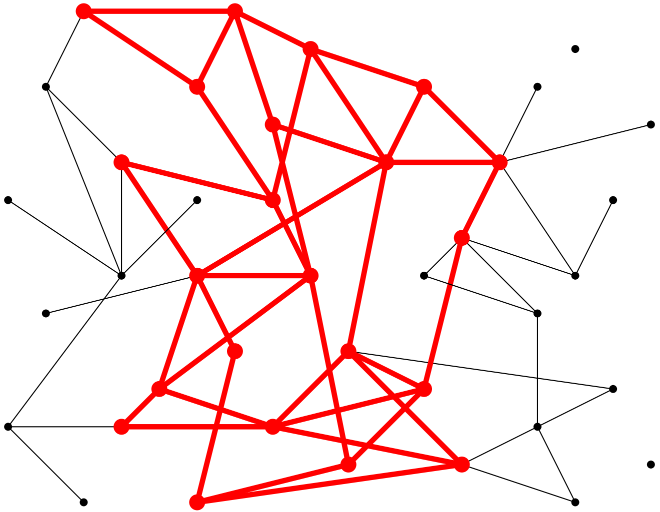

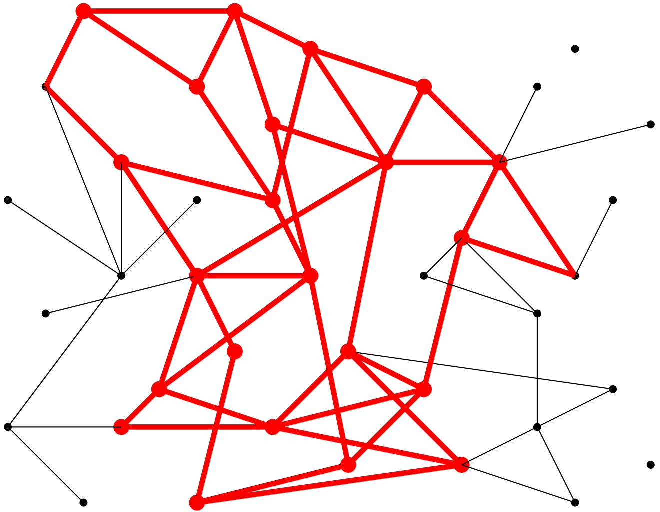

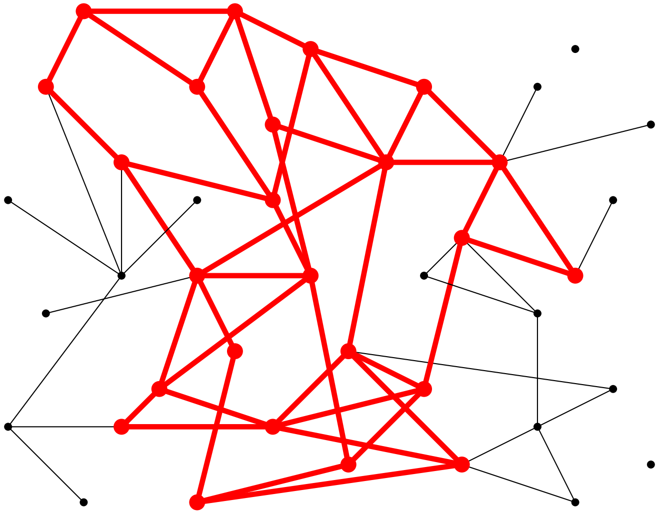

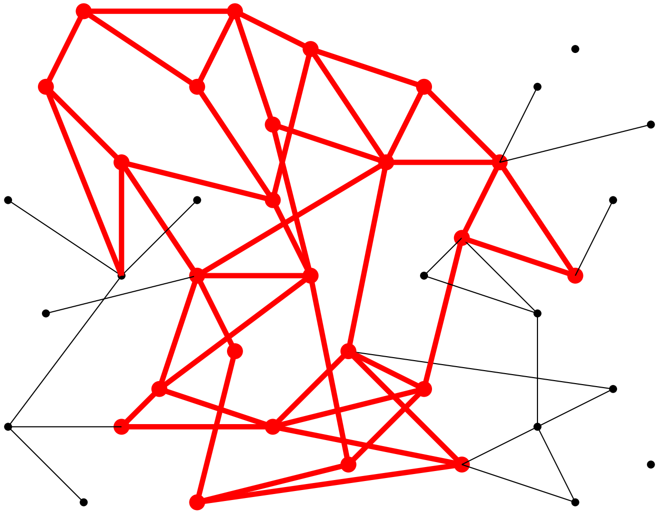

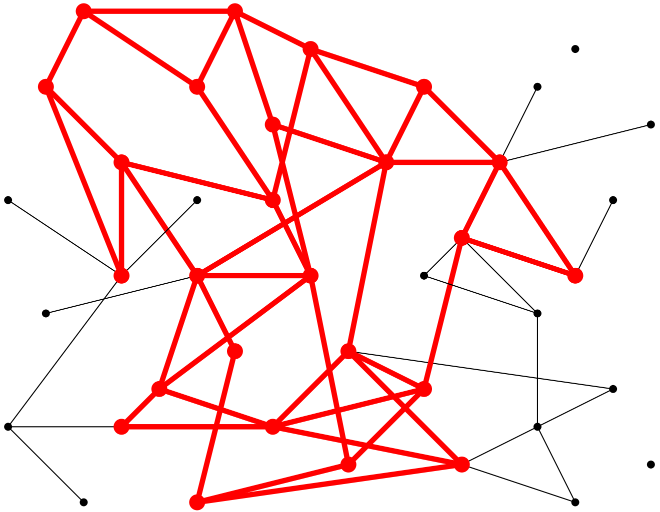

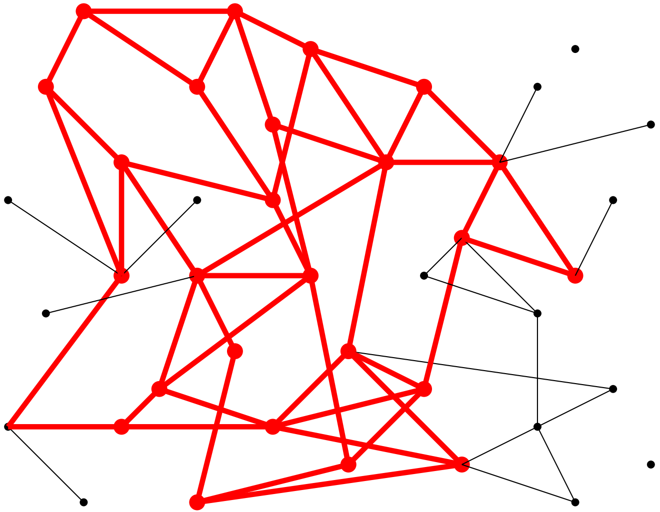

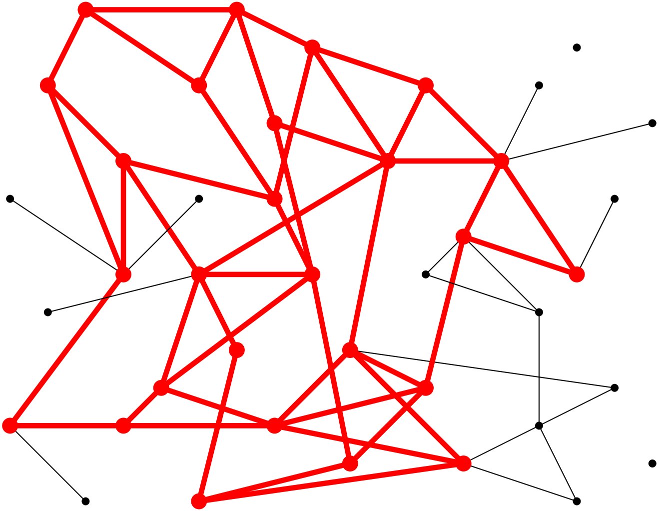

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1. $p = o\left(\frac{1}{n}\right)$ then $A_\infty = A_0\big(1 + o(1)\big)$ $\lim_{n\to\infty} \mathbb{P}\left\{A_\infty/A_0 > 1+\epsilon\right\} = 0$

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1. $p = o\left(\frac{1}{n}\right)$ then $A_\infty = A_0\big(1 + o(1)\big)$
2. $\frac{1}{n} \ll p \ll \frac{1}{\sqrt{n}}$ and let $A_0 = \frac{\alpha}{np^2}$
   - if $\alpha < \frac{1}{2}$ then $A_\infty < \frac{1}{np^2}$

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BLOODY MATHEMATICIANS!!!

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WHY DON'T THEY STUDY A MORE REALISTIC MODEL?

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EASIER SAID THAN DONE

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$$q_c([n]^d, k) = \left(\frac{\lambda(d,k) + o(1)}{\log_{(k-1)}(n)} \right)^{d-k+1}$$

by Balogh, Bollobás, Duminil-Copin, Morris

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Van Enter, talking about results on anisotropic bootstrap percolation:

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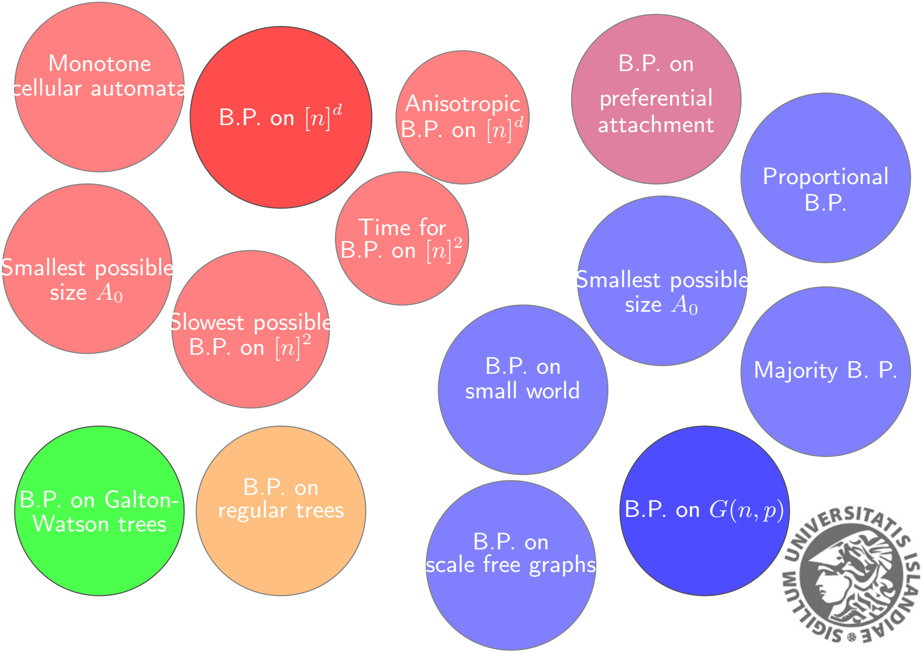
Van Enter, talking about results on anisotropic bootstrap percolation:

Numerically, that is for computational physicists e.g., these results are totally discouraging. Whereas in standard bootstrap percolation to obtain a 99 % accuracy in q_c *the order of magnitude of a square already needs to be of order* $O(10^{3000})$*, in the (1, 2)-model one needs to go even higher, namely to a doubly exponential size of order* $O\left(10^{10^{1400}}\right)$*. These findings illustrate the point made, that Cellular Automata, despite being discrete in state, space, and time, may still be ill-suited for computer simulations.*


Monotone cellular automata
B.P. on $[n]^d$
Anisotropic B.P. on $[n]^d$
B.P. on preferential attachment
Proportional B.P.
Smallest possible size A_0
Time for B.P. on $[n]^2$
Slowest possible B.P. on $[n]^2$
Smallest possible size A_0
B.P. on small world
Majority B. P.
B.P. on Galton-Watson trees
B.P. on regular trees
B.P. on scale free graphs
B.P. on $G(n, p)$