Modeling structure and function of complex networks and the brain

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NETWORKS

EURANDOM

Organization

▷ Lecture 1:
Complex Networks and the Brain.

▷ Lecture 2:
Complex Network Models and the Brain.

▷ Lecture 3:
Complex Networks and Brain Functionality.

Lecture 1:

Complex Networks and the Brain

- ▷ Complex Networks;
- ▷ Network statistics and Data;
- ▷ Abstract Math Perspective.

Networks of the brain

Several levels:

▷ Neuronal level: 10^{11} vertices of average degree 10^4;

▷ Functional level: much smaller, modular structure...

What is meaning network?

Features:

▷ Short time scales: stochastic process on network (non-linear?);

▷ Long time scales: network is changed by functionality brain (learning, pruning,...);

▷ Strong dependence between different regions network.

Big question:

What is a good network model for brain functionality?

Weighted brain graphs

▷ Brain: EEG data has weight or association (e.g., correlation between data signals) between any pair of vertices. Yields weighted complete graph.

Big question:

How to obtain informative network data from collection of weights?

Networks tend to be binary...

Thresholding?
Comparing networks with different average edge weights?
Union of smallest-weight paths?

▷ Weight distribution: Edge weights are likely dependent.

▷ Application to brain: Interpretation weights? Negative weights?

Random graphs in/and Brain

Kozma-Puljic (05):

There is dominant view that brains are not random and one should not use the term random graphs and networks for brains.

Without going into metaphysical debate, it can be safely assumed that brains, viewed either as complex deterministic machines or as random objects, can benefit from use of statistical methods in their characterization.

All models are wrong...

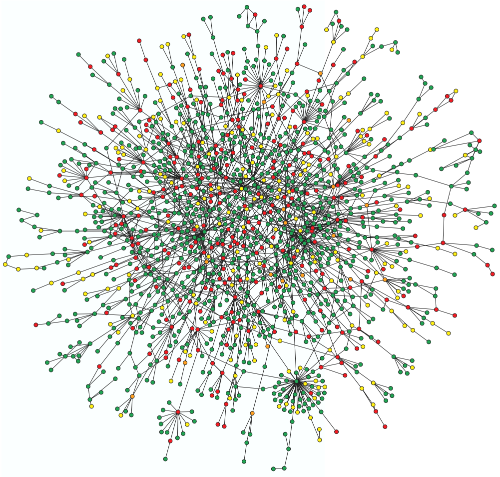
George Box (78):

Now it would be very remarkable if any system existing in the real world could be exactly represented by any simple model. However, cunningly chosen parsimonious models often do provide remarkably useful approximations....

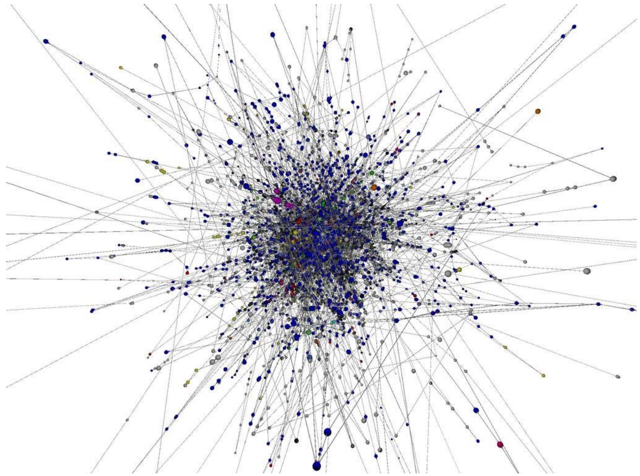
For such a model there is no need to ask the question “Is the model true?". If "truth" is to be the "whole truth" the answer must be "No". The only question of interest is "Is the model illuminating and useful?".

Question: How to “cunningly” choose a model for brain network topology and functionality?

Complex networks

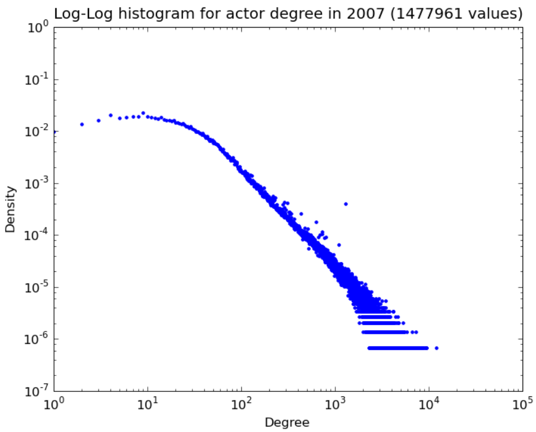


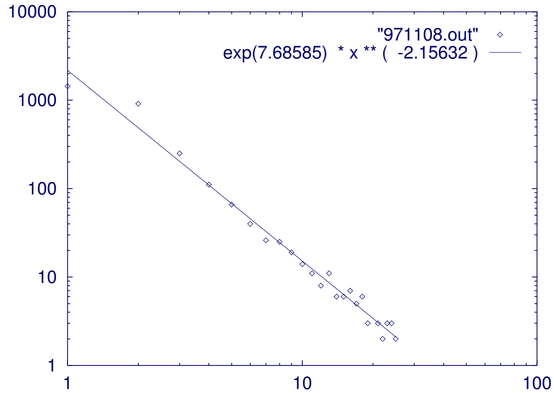
Yeast protein interaction network



Internet topology in 2001

Scale-free paradigm





Loglog plot of degree sequences in Internet Movie Data Base (2007)
and in the AS graph (FFF97)

Scale-free paradigm

Degree sequence $(N_1, N_2, N_3, \ldots)$ of graph:

N_1 is number of elements with degree 1,
N_2 is number of elements with degree 2,
...
N_k is number of elements with degree k.

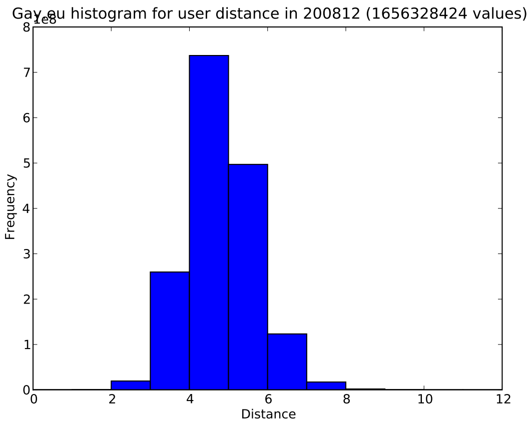
Then

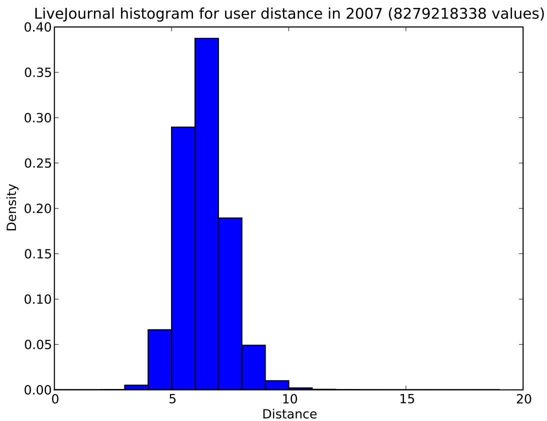
$$N_k \approx Ck^{-\tau},$$

precisely when

$$\log N_k \approx \log C - \tau \log k.$$

Small-world paradigm





Distances in social networks `gay.eu` on December 2008 and `livejournal` in 2007.

Network statistics

▷ Clustering:

$$C = \frac{3 \times \text{ number of triangles}}{\text{number of connected triplets}}.$$

Proportion of friends that are friends of one another.

▷ Assortativity:

$$\rho = \frac{\frac{1}{|E_n|}\sum_{ij\in E_n} d_i d_j - \left(\frac{1}{|E_n|}\sum_{ij\in E_n} d_i\right)^2}{\frac{1}{|E_n|}\sum_{ij\in E_n} d_i^2 - \left(\frac{1}{|E_n|}\sum_{ij\in E_n} d_i\right)^2}.$$

Correlation between degrees at either end of edge.

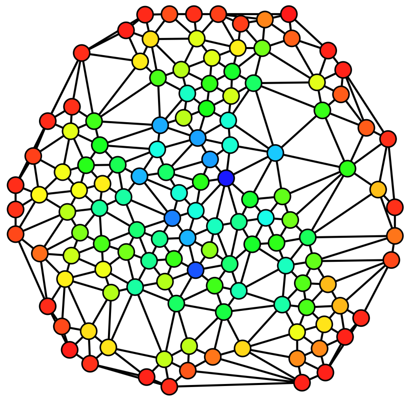
[Recent work vdH-Litvak (2013): flaws assortativity coefficient. Proposes rank correlations instead.]

Centrality measures

▷ Closeness centrality:
Measures to what extent vertex can reach others using few hops.
Vertices with low closeness centrality are central in network.

▷ Betweenness centrality:
Measures extent to which vertex connects various parts of network.

Betweenness large for bottlenecks.



▷ PageRank:
Measures extent to which vertex is visited by random walk.
Used in Google to rank importance in web pages.

Small-worldness

▷ Humphries-Gurney PLOS One (08). Let G be network and

$$\gamma_G^{\triangle} = \frac{C_G^{\triangle}}{C_{\mathrm{ER}}^{\triangle}}$$

describe clustering coefficient of G compared to ERRG, and

$$\lambda_G = \frac{L_G}{L_{\mathrm{ER}}}$$

describe average distances of G compared to ERRG.

Then, small-worldness parameter is

$$S_G^{\triangle} = \frac{\gamma_G^{\triangle}}{\lambda_G}.$$

▷ Not a big fan of this parameter: compares apples and pears...

Friendship paradox

Networking paradox (Scott L. Feld (1991)):

Why your friends have more friends than you do!

Wikipedia:

Twitter: The people a person follows almost certainly have more followers than they. This is because people are more likely to follow those who are popular than those who are not

Number of friends random individual is equal to

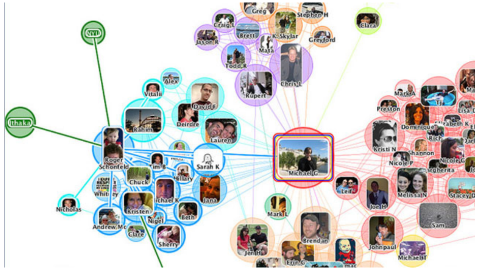
$$\mathbb{P}(D = k) = \frac{n_k}{n},$$

where n_k is number of vertices with degree k and n is network size.

Friendship paradox

Average number of friends random individual equals

$$\sum_k k\mathbb{P}(D = k) = \frac{2|E|}{n},$$



where $|E|$ is number of edges.

Wikipedia: The average number of friends that a typical friend has can be modeled by choosing, uniformly at random, an edge of the graph and an endpoint of that edge, and again calculating the degree of the selected endpoint.

With $D^\star$ degree vertex in random edge,

$$\mathbb{E}[D^\star] = \mathbb{E}[D] + \frac{\mathrm{Var}(D)}{\mathbb{E}[D]} > \mathbb{E}[D].$$

Your friends have more friends than you do!

Friendship paradox

Take vertex uniformly at random, then take one of its neighbors and inspect its degree. Denote degrees at both sides (D_1, D_2). Then,

- $\triangleright$ D_1 has same distribution as D, but
- $\triangleright$ D_2 does not have same distribution as $D^\star$!

Still [Theorem 1.2], except when all degrees are equal,

$$\mathbb{E}[D_2] > \mathbb{E}[D]!$$

Your friends have more friends than you do!

Empirical brain networks

Complex brain networks: graph theoretical analysis of structural and functional systems

Bullmore and Sporns, Nature Reviews (09):

Empirical brain networks consistently show following features:

- ▷ Small worlds;
- ▷ High clustering;
- ▷ Hub-like degree structure, with (possibly exponentially truncated) power-law degree sequences;
- ▷ Hubs have rich-club organisation;
- ▷ Modular structure;
- ▷ Long-range spatial connections occurring at low rate.

Networks change with age.

Why network analysis?

Complex brain networks: graph theoretical analysis of structural and functional systems

Bullmore and Sporns, Nature Reviews (09):

Emprirical analysis shows that network topology is affected in patients with mental disorders:

Disorders investigated include Alzheimer disease (AD) and schizophrenia:

▷ Small world nature diminished, suggesting loss of efficiency of brain functionality;

▷ Clustering affected by AD, most often lower.

Lecture 2:

Complex Network Models and the Brain

- ▷ Random Graphs;
- ▷ Network Models;
- ▷ Properties Models;
- ▷ Applicability to the Brain.

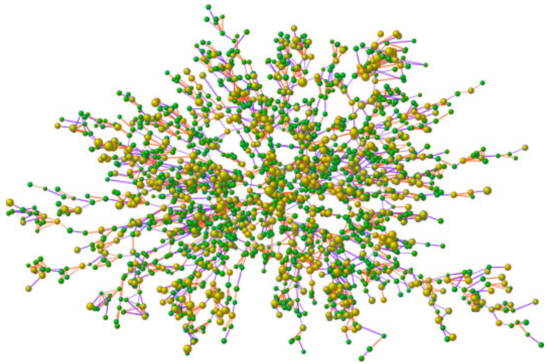
Modeling networks

Use random graphs to model uncertainty in formation

connections between elements.

▷ Static models:
Graph has fixed number of elements:

Configuration model



▷ Dynamic models:
Graph has evolving number of elements:

Preferential attachment model

Many models!

Universality??

Erdős-Rényi

Vertex set $[n] := \{1, 2, \ldots, n\}$.

Erdős-Rényi random graph is random subgraph of complete graph on $[n]$ where each of $\binom{n}{2}$ edges is occupied with probab. p.

Simplest imaginable model of a random graph.

$\triangleright$ Attracted tremendous attention since introduction 1959, mainly in combinatorics community.

Probabilistic method (Erdős et al).

$\triangleright$ Egalitarian: Every vertex has equal connection probabilities. Misses hub-like structure of real networks.

Inhomogeneous versions have been suggested and investigated.

Null model 1

Many adaptations, including the original Erdős-Rényi random graph, where a fixed number m of edges is chosen uniformly at random without replacement.

Models are closely related when taking

$$p \approx 2m/(n(n-1)).$$

Random graph with fixed number of edges is uniform random graph with that number of edges:

Null model.

Yields bench mark to compare real-world networks to having same number of edges.

$\triangleright$ Directed ERRG: One can also study directed versions.

Configuration model

$\triangleright$ Invented by Bollobás (1980)
to study number of graphs with given degree sequence.
Inspired by Bender+Canfield (1978)
Giant component: Molloy, Reed (1995)
Popularized by Newman, Strogatz, Watts (2001).

$\triangleright$ n number of vertices;
$\triangleright$ $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$ sequence of degrees is given.

Often take $(d_i)_{i\in[n]}$ to be sequence of independent and identically distributed (i.i.d.) random variables with certain distribution.

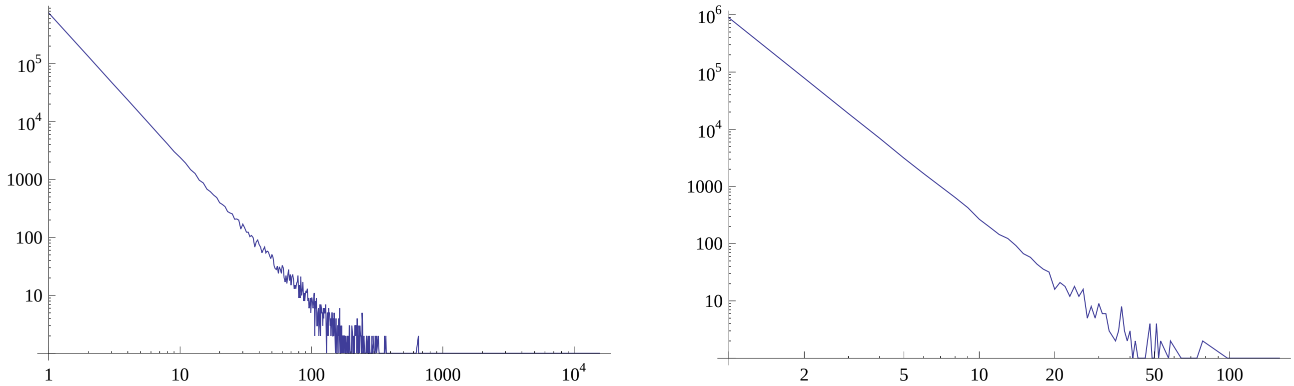
$\triangleright$ Special attention to power-law degrees, i.e., for $\tau > 1$ and c_τ

$$\mathbb{P}(d_1 \geq k) \approx c_\tau k^{-\tau+1}.$$

Power-laws CM

▷ Special attention to power-law degrees, i.e., for $\tau > 1$ and c_τ

$$\mathbb{P}(d_1 \geq k) = c_\tau k^{-\tau+1}(1 + o(1)).$$



Loglog plot of degree sequence CM with i.i.d. degrees
$n = 1{,}000{,}000$ and $\tau = 2.5$ and $\tau = 3.5$, respectively.

Graph construction

$\triangleright$ Assign d_j half-edges to vertex j. Assume total degree

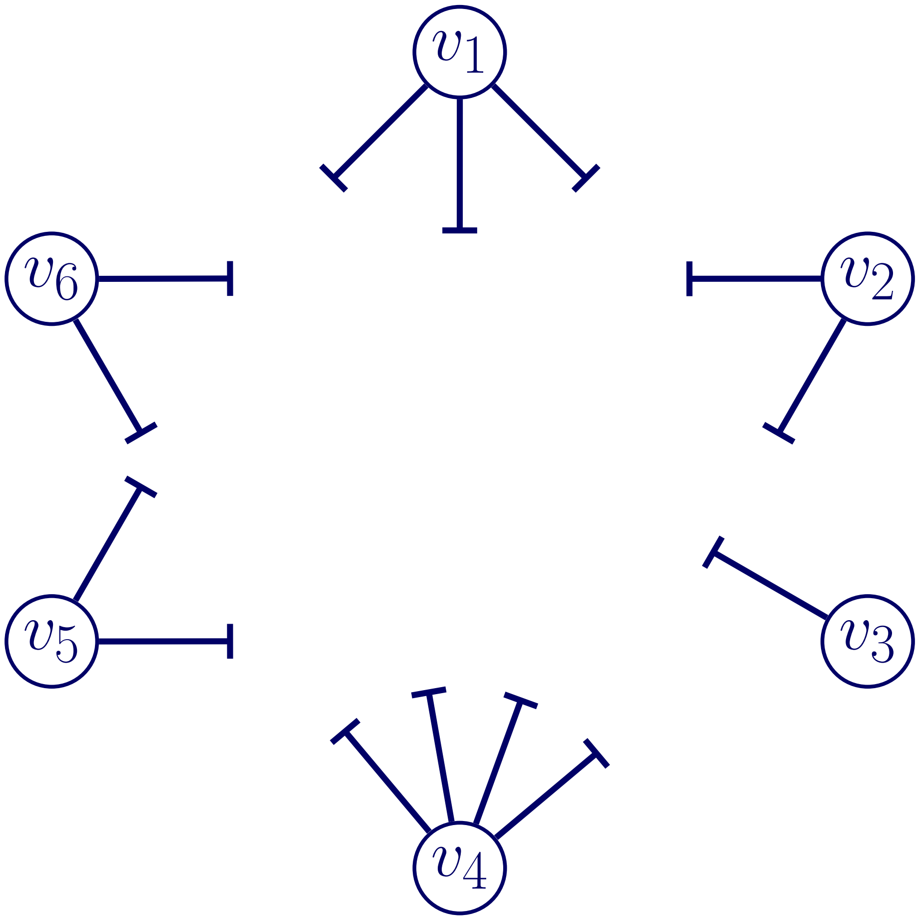
$$\ell_n = \sum_{i \in [n]} d_i$$

is even.

$\triangleright$ Pair half-edges to create edges as follows:
Number half-edges from 1 to ℓ_n in any order.
First connect first half-edge at random with one of other $\ell_n - 1$ half-edges.

$\triangleright$ Continue with second half-edge (when not connected to first) and so on, until all half-edges are connected.

$\triangleright$ Resulting graph is denoted by $\mathrm{CM}_n(\boldsymbol{d})$.



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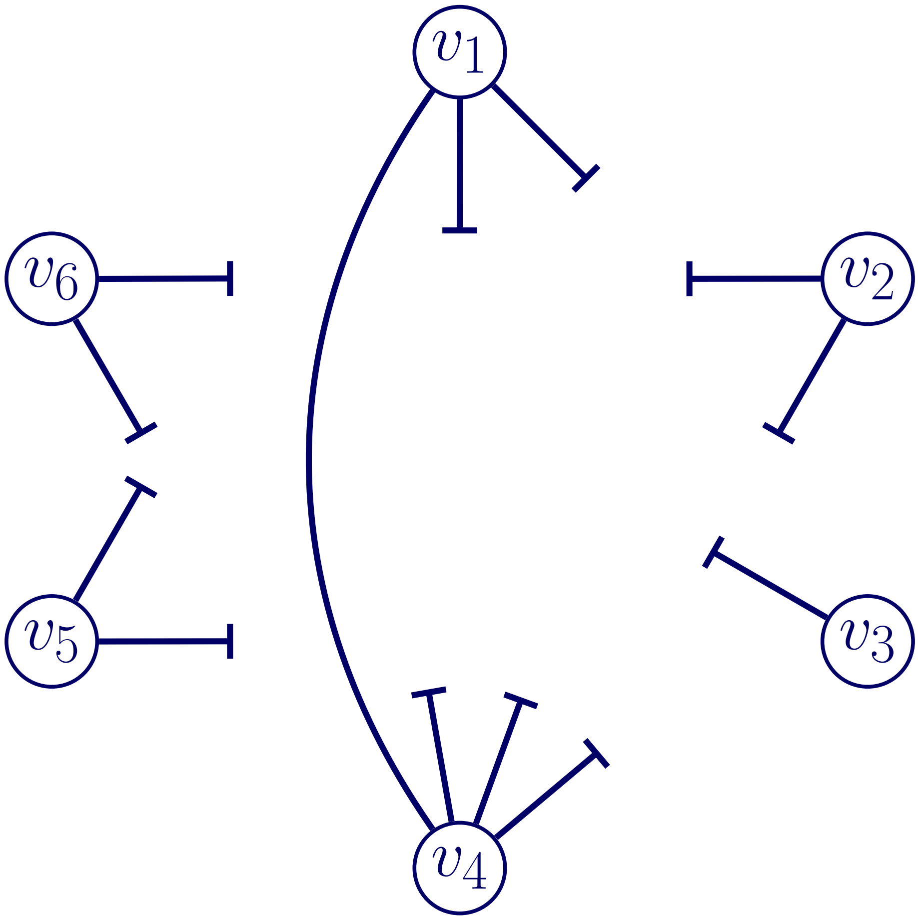
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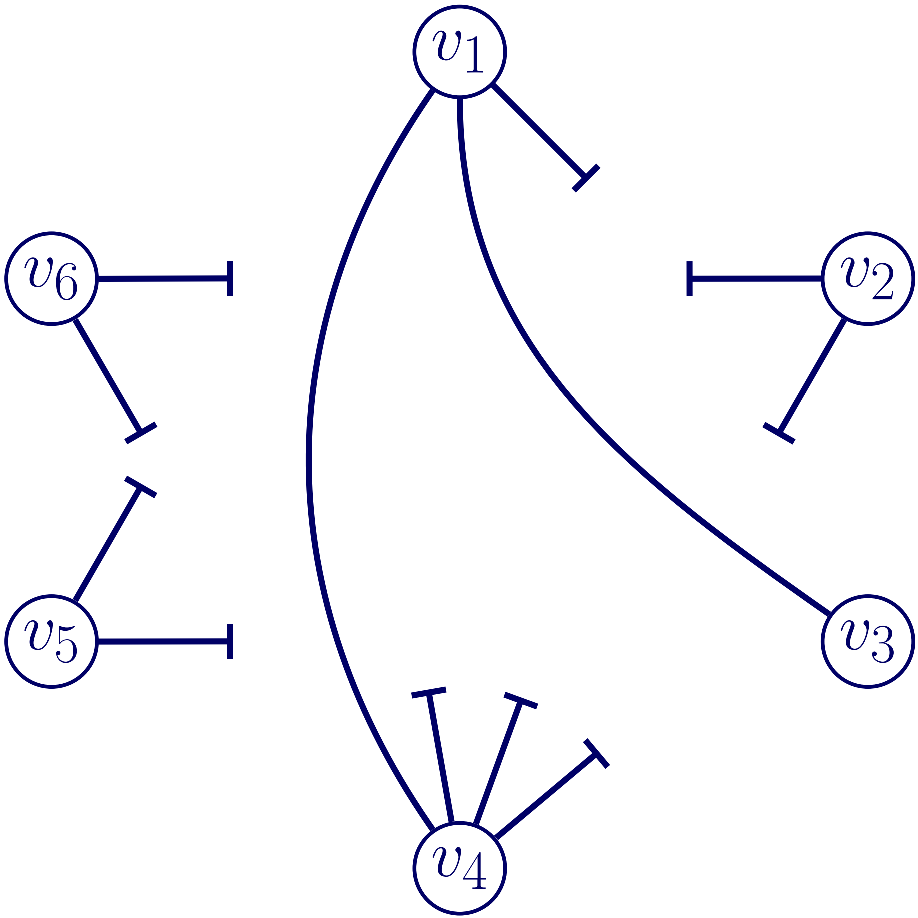
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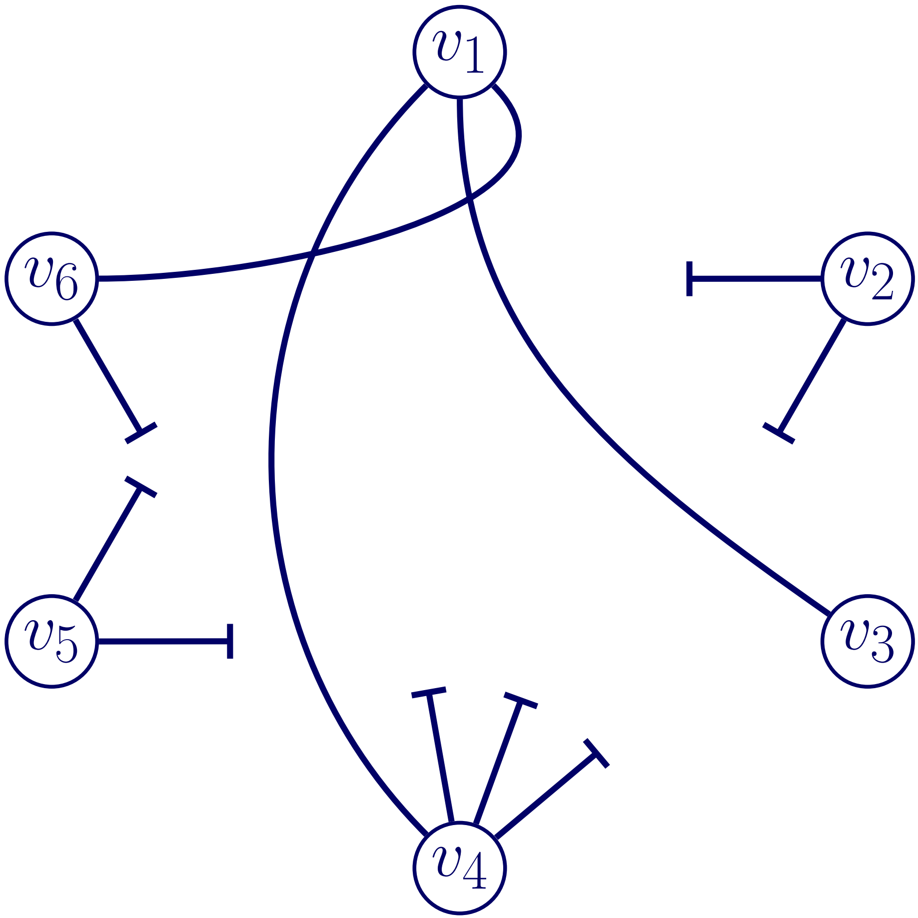
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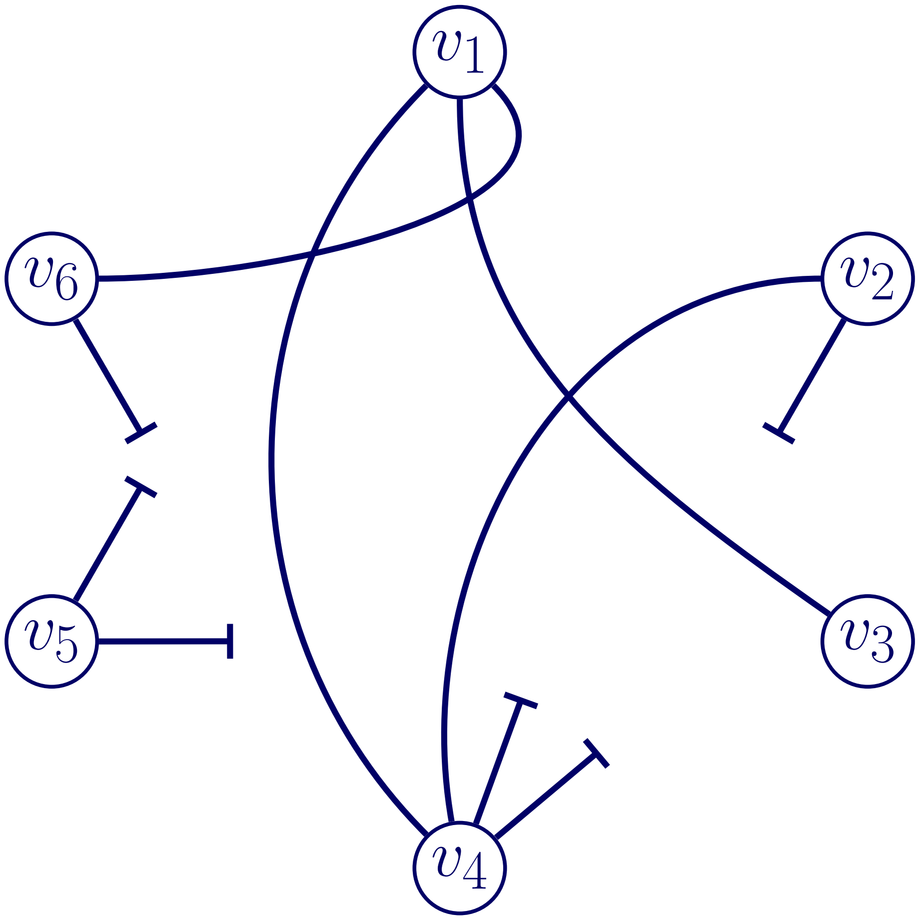
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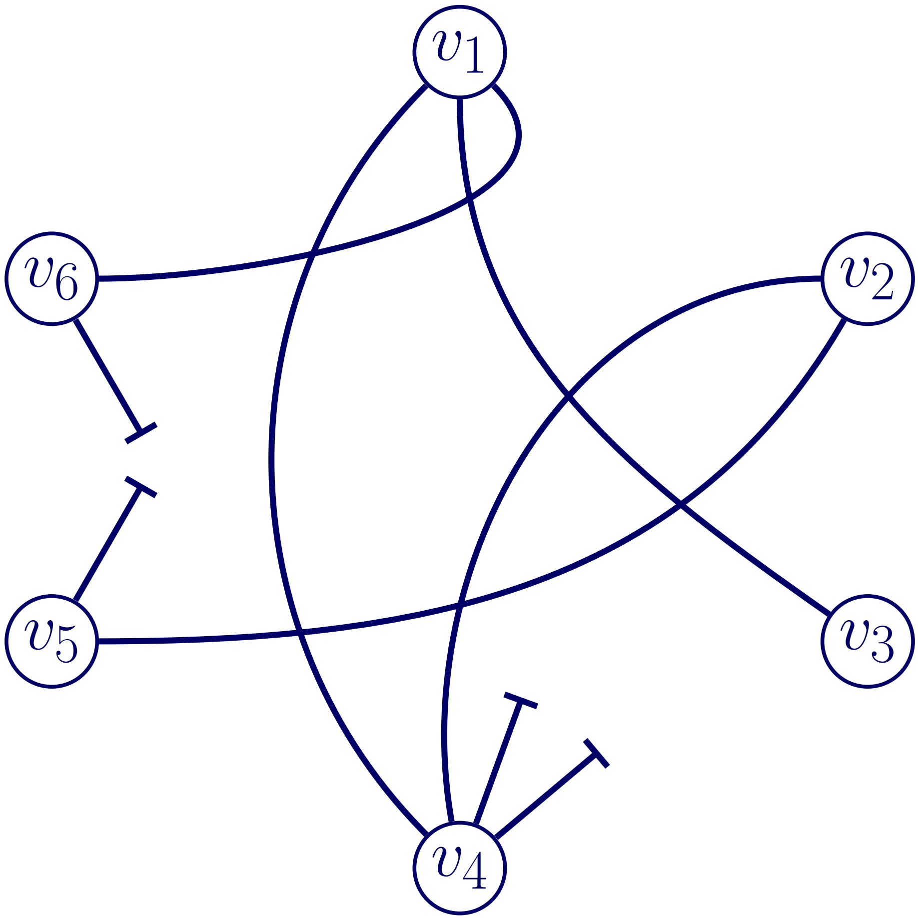
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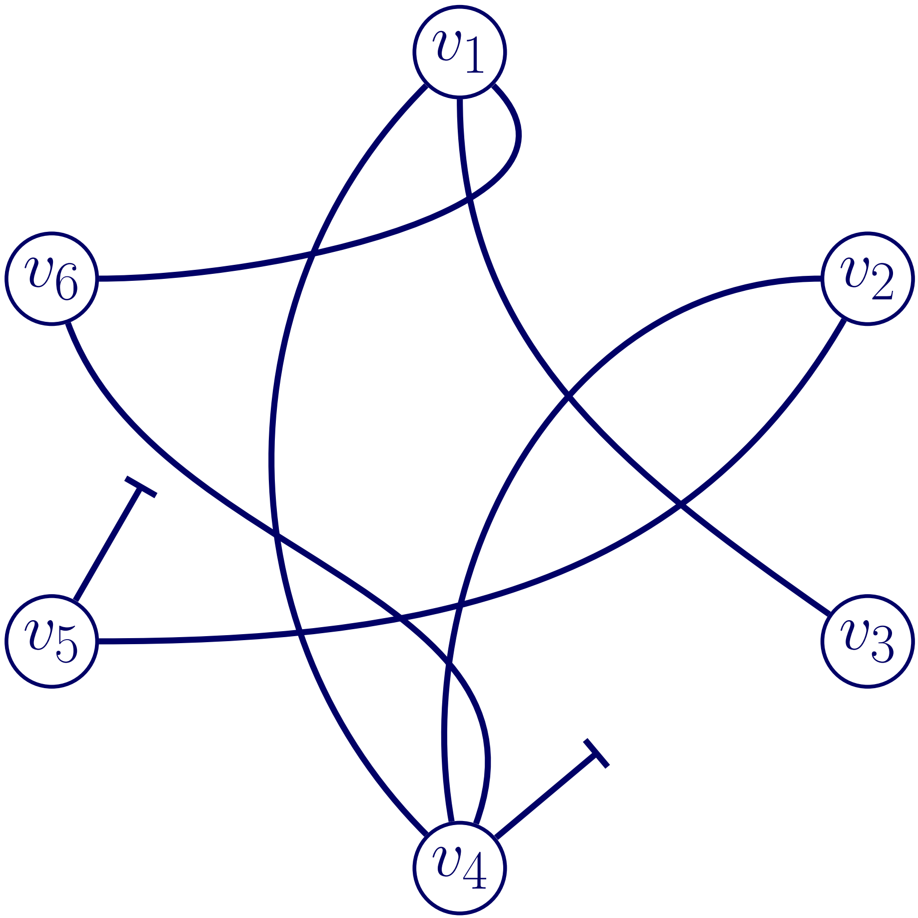
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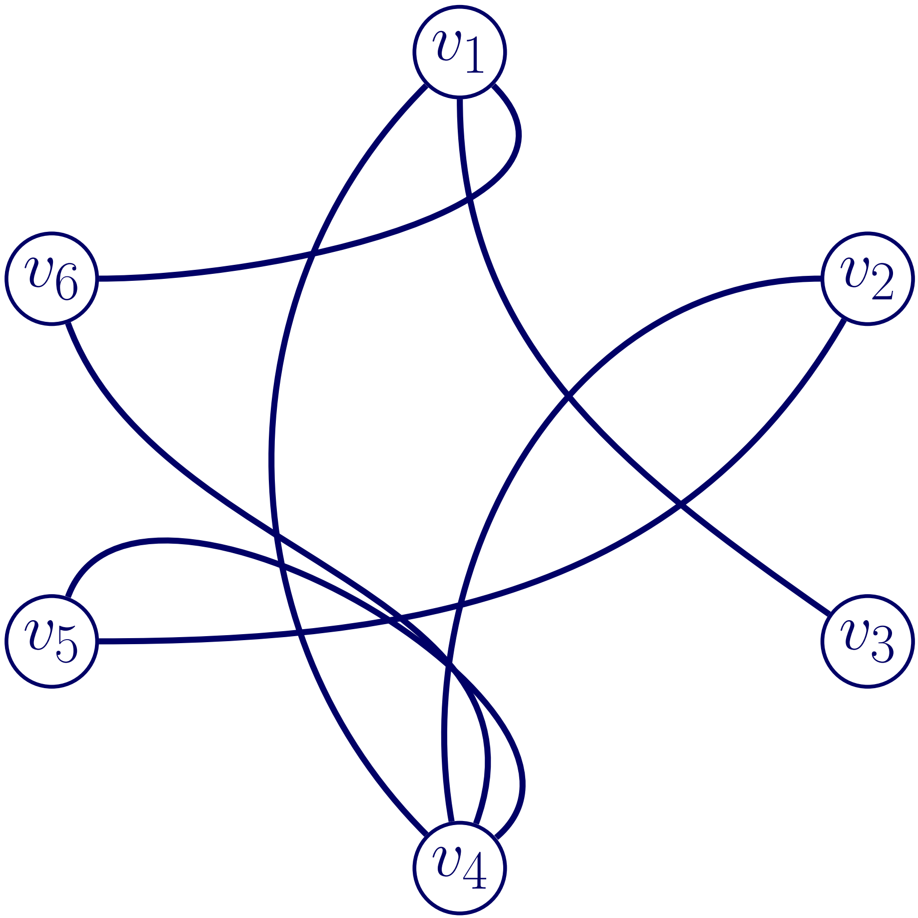

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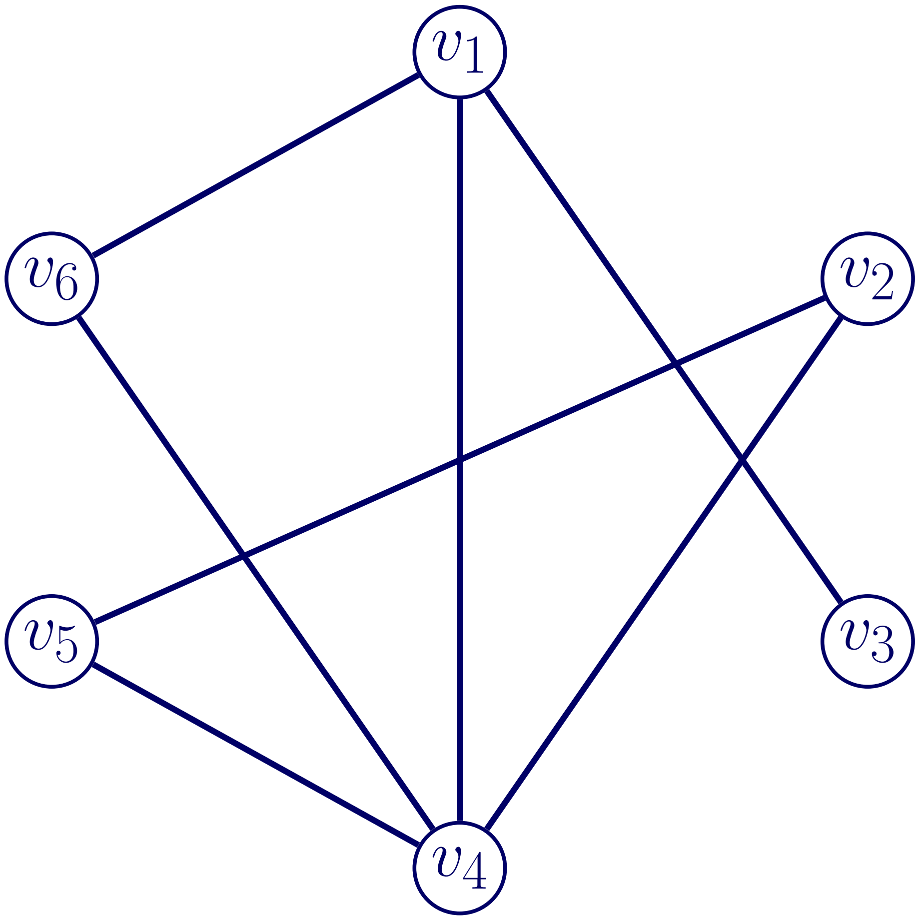
v_1
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Null model 2

Configuration model with fixed degrees and conditioned on simplicity yields uniform random graph with those degrees:

Null model.

Yields bench mark to compare real-world networks.

When degrees are not too heavy-tailed,

Probability simplicity uniformly positive.

$\triangleright$ Note: degrees in ERRG are close to Poisson, which does not fit well with many real-world networks.

$\triangleright$ Can also create uniform random graph with prescribed degrees by rewiring edges from any simple graph with those degrees. Is practical way to simulate graph. Problem: mixing time is unknown.

$\triangleright$ Directed CM: Many results easily extend.

Graph distances in CM

H_n is graph distance between uniform pair of vertices in graph.

Theorem 1. (vdHHVM03). When $\nu = \mathbb{E}[D(D-1)]/\mathbb{E}[D] \in (1,\infty)$ and $\mathbb{E}[D_n^2] \to \mathbb{E}[D^2]$, conditionally on $H_n < \infty$,

$$\frac{H_n}{\log_\nu n} \xrightarrow{\mathbb{P}} 1.$$

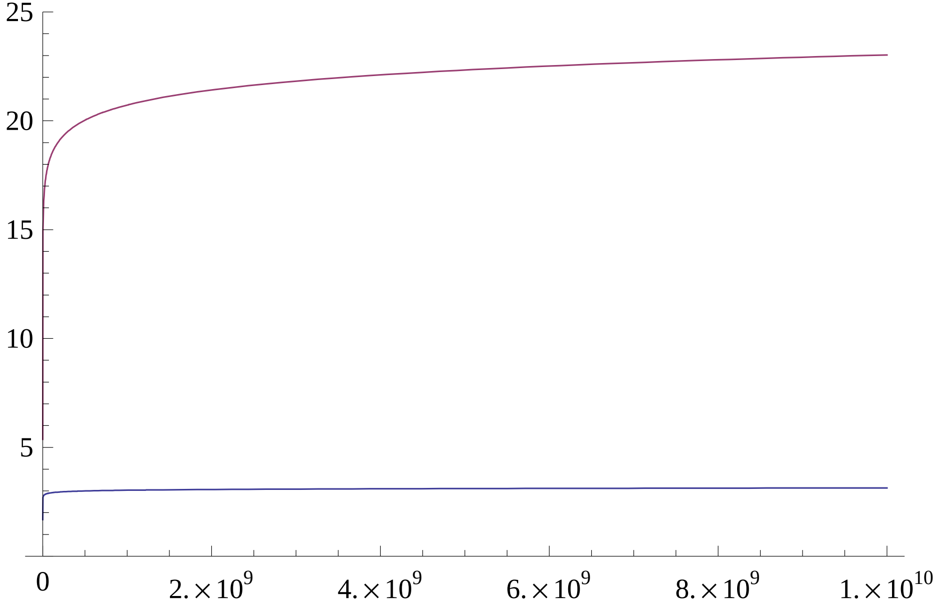
For i.i.d. degrees having power-law tails, fluctuations are bounded.

Theorem 2. (vdHHZ07, Norros+Reittu 04). When $\tau \in (2,3)$, conditionally on $H_n < \infty$,

$$\frac{H_n}{\log\log n} \xrightarrow{\mathbb{P}} \frac{2}{|\log(\tau-2)|}.$$

For i.i.d. degrees having power-law tails, fluctuations are bounded.

$x \mapsto \log\log x$ grows extremely slowly



Plot of $x \mapsto \log x$ and $x \mapsto \log\log x$.

Preferential attachment

Albert-Barabási (1999):
Emergence of scaling in random networks (Science).

23020 cit. (23-7-2015).

Bollobás, Riordan, Spencer, Tusnády (2001):
The degree sequence of a scale-free random graph process (RSA)

[In fact, Yule 25 and Simon 55 already introduced similar models.]

In preferential attachment models, network is growing in time, in such a way that new vertices are more likely to be connected to vertices that already have high degree.

Rich-get-richer model.

Preferential attachment

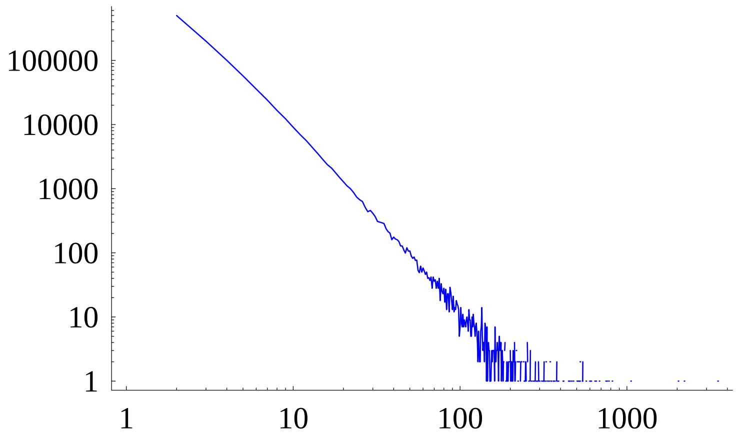
At time n, single vertex is added with m edges emanating from it. Probability that edge connects to i^{th} vertex is proportional to

$$D_i(n-1)+\delta,$$

where $D_i(n)$ is degree vertex i at time n, $\delta > -m$ is parameter.

Yields power-law degree sequence with exponent $\tau = 3 + \delta/m > 2$.

BRST01 $\delta = 0$,
DvdEvdHH09,...



$m = 2, \delta = 0, \tau = 3, n = 10^6$

Distances PA models

Theorem 3 (Bol-Rio 04). For all $m \geq 2$ and $\tau = 3$,

$$H_n = \frac{\log n}{\log \log n}(1 + o_{\mathbb{P}}(1)).$$

Theorem 4 (Dommers-vdH-Hoo 10). For all $m \geq 2$ and $\tau \in (3, \infty)$,

$$H_n = \Theta(\log n).$$

Theorem 5 (Dommers-vdH-Hoo 10, DerMonMor 11). For all $m \geq 2$ and $\tau \in (2, 3)$,

$$\frac{H_n}{\log \log n} \xrightarrow{\mathbb{P}} \frac{4}{|\log(\tau - 2)|}.$$

Network modeling mayhem

Models:

▷ Configuration Model
▷ Inhomogeneous Random Graphs
▷ Preferential Attachment Model

What is bad about these models?

▷ Low clustering and few short cycles (unlike social networks);
▷ No communities (unlike collaboration networks and WWW);
▷ No attributes (geometry, gender,...);

Models are caricature of reality!

Network models I

▷ **Configuration model with clustering:**
Input per vertex i is number of simple edges, number of triangles, number of squares, etc. Then connect uniformly at random.

Result: Random graph with (roughly) specified degree, triangle, square, etc distribution over graph.

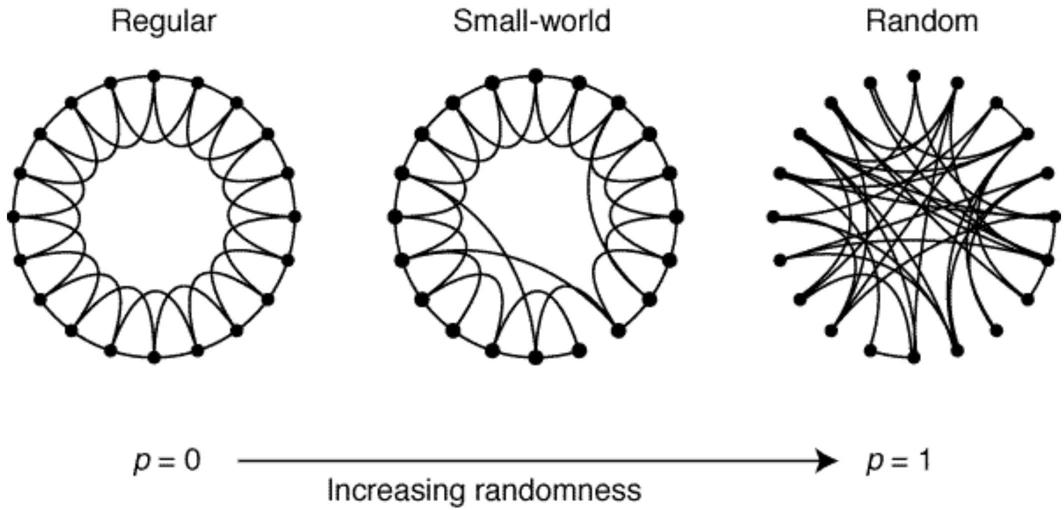
Application: Social networks?

▷ **Small-world model:**
Start with d-dimensional torus (=circle $d = 1,$ donut $d = 2,$ etc).
Put in nearest-neighbor edges. Add few edges between uniform vertices, either by rewiring or by simply adding.

Result: Spatial random graph with high clustering, but degree distribution with thin tails.

Application: None? Often used for brain.

Small-world model



Network models II

▷ **Random intersection graph:**
Specify collection of groups. Vertices choose group memberships. Put edge between any pairs of vertices in same group.

Result: Flexible collection of random graphs, with high clustering, communities by groups, tunable degree distribution.

Application: Collaboration graphs?

▷ **Spatial preferential attachment model:**
First give vertex uniform location. Let it connect to close by vertices with probability proportionally to degree.

Result: Spatial random graph with scale-free degrees and high clustering.

Application: Social networks, WWW?

Brain network I

▷ Generative models of the human connectome

Betzel, de Reus, Hagmann, van den Heuvel, Sporns et al (2015).

Dynamical network: Start with sparse seed network;
At each time step, choose unconnected vertices u, v and update by forming edge between nodes u, v with probability

$$P(u, v) = E(u, v)^{\eta} K(u, v)^{\gamma},$$

where
▷$E(u, v)$ denotes Euclidean distance brain regions u and v;
▷η controls characteristic connection length;
▷$K(u, v)$ represents relationship between nodes u and v;
▷γ scales relative importance of $K(u, v)$.

Examples of $K(u, v)$ include
$K(u, v) = d_u d_v$ giving variant of preferential attachment;
$K(u, v) = |d_u - d_v|$ giving disassortative networks when $\gamma > 0$;

Brain network I

$\triangleright$ Generative models of the human connectome

Betzel, de Reus, Hagmann, van den Heuvel, Sporns et al (2015).

Best fit for matching rule, where

$$K(u,v) = \frac{|\Gamma(u) \cap \Gamma(v)|}{|\Gamma(u) \cup \Gamma(v)|},$$

where $\Gamma(u), \Gamma(v)$ are neighborhood sets of u, v.

Here estimation yields $\hat{\eta} = -0.98 \pm 0.37$, and $\hat{\gamma} = 0.42 \pm 0.04$.

$\triangleright$ Kolmogorov-Smirnov statistics quantifying discrepancy between synthetic and observed graphs in terms of degree, clustering, betweenness centrality, and edge length distributions are all ≈ 0.10.

Brain network II

▷ Modelling the development of cortical systems networks

Kaiser and Hilgetag, Neurocomputing 58-60 (2004) 297–302.

Dynamical network: Start with empty network;
At each step new area added until reaching target number nodes.
New area generated at random position of embedding space.

Probability of connection between new area u and existing area v

$$P(u, v) = \beta \mathrm{e}^{-\alpha d(u,v)},$$

with $d(u, v)$ is distance between nodes, β, α scaling coefficients.

Comparison on basis of clustering coefficient and average shortest path length gives good match.

Scale-free percolation

▷ **Scale-free percolation:**
Vertex set $\mathbb{Z}^d$. Each vertex x has a weight W_x, which form a collection of independent and identically distributed random variables.

Put edge between x and y with probability, conditionally on weights, equal to

$$p_{xy} = 1 - \mathrm{e}^{-W_x W_y / \|x-y\|^{\alpha}},$$

where $\alpha > 0$ is parameter model.

Result: Spatial random graph with scale-free degrees when weights obey power-law, high clustering and small-world.

Application: Social networks, WWW, brain?

Distances other models

Similar results (though often weaker) proved for related models:
$\triangleright$ Random intersection graphs;
$\triangleright$ Small-world model;
$\triangleright$ Scale-free percolation.

Full extent of universality paradigm still unclear.

Work in progress!

Lecture 3:

Complex Networks and Brain Functionality

- ▷ Stochastic Processes for Functionality;
- ▷ Excitation vs. Inhibition;
- ▷ Ising model, Bootstrap Percolation, Integrate and Fire Models;
- ▷ Applicability to the Brain.

Weighted brain graphs

▷ Brain: EEG/fMRI data has weight or association (e.g., correlation between data signals) between any pair of vertices. Yields weighted complete graph.

Big question:

How to obtain informative network data from collection of weights?

Networks tend to be binary...

Thresholding?
Comparing networks with different average edge weights?
Union of smallest-weight paths?

▷ Weight distribution: Edge weights are likely dependent.

▷ Application to brain: Interpretation weights? Negative weights?

All models are wrong

George Box (78): Now it would be very remarkable if any system existing in the real world could be exactly represented by any simple model. However, cunningly chosen parsimonious models often do provide remarkably useful approximations....

For such a model there is no need to ask the question “Is the model true?". If "truth" is to be the "whole truth" the answer must be "No". The only question of interest is "Is the model illuminating and useful?".

Question: How to “cunningly” choose a model for brain topology and functionality?

Brain functionality

Mathematical models tend to focus on excitatory interactions. For global stability, particularly inhibitory interactions are crucial.
Will describe excitatory models studied in math/neuroscience, as well as their inhibitory extension.

$\triangleright$ General framework: Let $G = (V, E)$ denote graph.

$\triangleright$ Functionality: We view brain functionality as (stochastic) process living on network. External stimuli cause reaction from system. This is simple model for brain functionality.

> Reality is much harder! Network brain changes due to its functionality.

Ising Model

Ising model is spin system, where vertices can be in two states $\{-1, 1\}$. Invented as model for magnetism. For brain, think of 1 indicating that neuron fires, -1 as neuron not firing.

Ising model aims to describe collection of firing neurons using Boltzman distribution: Let $G = (V, E)$ denote graph, and $\sigma = (\sigma_i)_{i \in V}$ as collection of firing neurons. Then, distribution of firing neurons equals

$$\mu_{\beta,h}(\sigma) = \frac{1}{Z(\beta, h)} \mathrm{e}^{-H(\sigma)},$$

where $H(\sigma)$ is Hamiltonian of configuration of firing neurons σ

$$H(\sigma) = -\beta \sum_{(x,y) \in E} \sigma_x \sigma_y - h \sum_{x \in V} \sigma_x.$$

and $Z(\beta, h)$ is normalization constant. Here $\beta > 0$ determines preference for neurons to fire together, and $h \in \mathbb{R}$ determines likeliness of firing.

Ising model dynamics

At time $n \geq 0$, one randomly chosen neuron v changes his/her firing status σ_v to $1 - \sigma_v$ with probability

$$\min\left\{ \mathrm{e}^{-[H(\sigma^v) - H(\sigma)]}, 1 \right\},$$

where σ^v is obtained from σ by flipping firing status of v. Measure $\mu_{\beta,h}$ is stationary distribution.

Phase transition:
Exists critical β_c. Below it stationary distribution is unique for $h = 0$, Above it, two distinct stationary distributions (obtained from configurations where all firing statusses agree):

Positive instantaneous magnetization.

Substantial knowledge of near-critical behavior of Ising model on random graphs.

Ising phase transition CM

▷ Critical value:

$$\begin{array}{rcl} \beta_c = 0 & \text{for} & \tau \in (2,3), \\ \beta_c = \operatorname{atanh}(1/\nu) & \text{for} & \tau > 3,\ \nu > 1. \end{array}$$

Here $\nu = \mathbb{E}[D(D-1)]/\mathbb{E}[D] < \infty$ when $\tau > 3$ or $\mathbb{E}[D^2] < \infty$.

Means that, for $\tau \in (2,3)$, tiny external effects (media/sensory system?) can cause opinion population to flip.

▷ Computation of various limits as graph grows large, such as average spin or magnetization:

$$M(\beta, h) = \lim_{n\to\infty} \frac{1}{n} \sum_{i\in[n]} \sigma_i.$$

Critical exponents CM

For (β, h) close to $(\beta_c, 0)$,

$$M(\beta_c, h) \sim h^{1/\delta}, \qquad\qquad M(\beta, 0^+) \sim (\beta - \beta_c)^{\beta},$$

where

$$(\delta, \beta) = (3, 1/2) \qquad\qquad \text{for } \tau > 3;$$

while

$$(\delta, \beta) = (\tau - 2, 1/(\tau - 3)) \qquad \text{for } \tau \in (3, 5).$$

Challenges and questions:

- ▷ Can we apply Ising model to brain?
- ▷ Deal with hierarchical modularity;
- ▷ Role hubs...

Ising and fMRI

Ising-like dynamics in large-scale functional brain networks

Daniel Fraiman, Balenzuela, Foss and Chialvo (Phys Rev E 09):

Interesting comparison of 2D Ising model and

fMRI data of brain resting state.

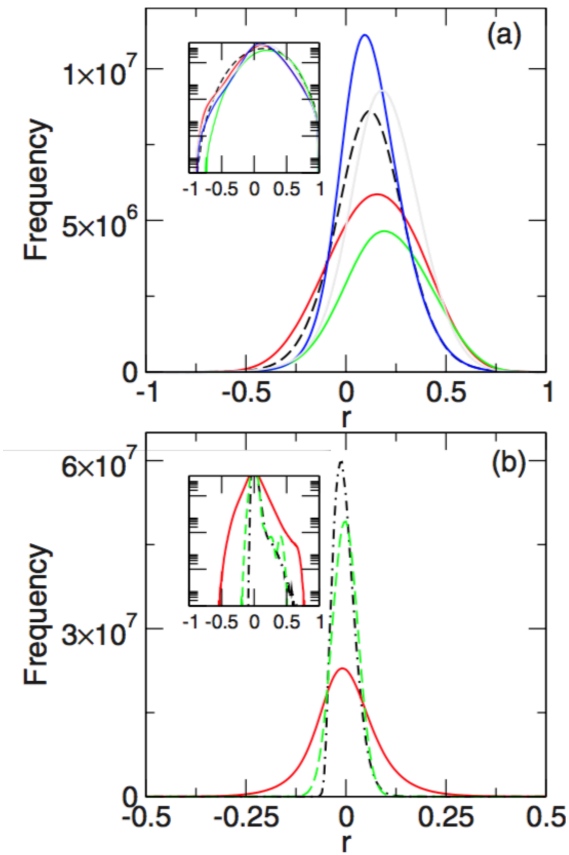
$64 \times 64 \times 49$ sites corresponding to voxels of dimension $3.4375 \times 3.4375 \times 3\ \mathrm{mm}^3$.

Compared measured correlations to correlations measured in dynamical Ising model at

critical temperature.

Reasonable comparison.

Ising-fMRI



(a)
Frequency
1×10^7
5×10^6
0
-1
-0.5
0
0.5
1
r
-1 -0.5 0 0.5 1
(b)
Frequency
6×10^7
3×10^7
0
-0.5
-0.25
0
0.25
0.5
r
-1 -0.5 0 0.5 1

Bootstrap percolation

Bootstrap percolation is dynamical process of how firing of neurons sweeps through network:

$\triangleright$ Choose initial random set of excited neurons;
$\triangleright$ When non-excited neuron has at least k excited neighbors, it will become excited in next phase;
$\triangleright$ Iterate dynamics indefinitely.

Phase transition:
Exists critical number $\boldsymbol{a}_c$. When fewer than $\boldsymbol{a}_c$ random vertices initially excited, excitation does not reach all vertices, while above it, end set of excited neurons is (almost) entire graph.

Can do this in discrete time (rounds) or continuous time.

Bootstrap percolation and Inhibition

Brain: when signal activates small part of local ensemble of neurons, activity spreads through to recurrent connections.

But only up to point where inhibitory neurons are strong enough to stop spread activitation: Very different input strengths lead to similar levels of activity never surpassing certain upper bound.

Effect observed experimentally: Heeger (92) calls it

normalization of cell responses.

▷ Discrete-time process has weird non-monotone behavior.

Bootstrap percolation and Inhibition

Einarsson, Lengler, Mousset, Panagiotou, Steger (preprint 2015):

▷ Continuous-time process:
τ denotes proportion of inhibitory neurons, $1-\tau$ proportion of excitatory neurons. Each directed edge with excitatory (inhibitory) origin independently present with probability p (resp. $\gamma p.$)

Take $1/n \ll p \ll n^{-1/k}$, and activation spreads when excitatory active neighbors exceeds inhibitory neighbors by k. Then

(i) If $\tau < 1/(1+\gamma)$, process almost percolates;
(ii) If $\tau \geq 1/(1+\gamma)$, and if $a \geq \boldsymbol{a}_c = o(n)$, then whp there are $(1-\tau)^k n/(\gamma\tau)^k + o(n)$ active vertices at final time. Here

$$\boldsymbol{a}_c = (1-1/k)\left(\frac{(k-1)!}{(1-\tau)^k np^k}\right)^{1/(k-1)}.$$

Neuropercolation

▷ Modeling scale-free neurodynamics using neuropercolation approach

Kozma and Puljic (2007).

▷ Random graph theory and neuropercolation for modeling brain oscillations at criticality

Kozma and Puljic (2015).

▷ Phase transitions in the neuropercolation model of neural populations with mixed local and non-local interactions

Kozma, Puljic, Balister, Bollobás and Freeman (2004).

Neuropercolation is generalization of bootstrap percolation:
Allows random transitions: back to inactive and inhibition.

Models dynamical behavior of neuropil = densely interconnected neural tissue in cortex.

Neuropercolation

Activity of at time t in vertex x is denoted by $a_t(x) \in \{0, 1\}$.
Denote event C_t by

$$C_t = \{ \sum_{x \in \Lambda(x)} a_t(x) \leq |\Lambda(x)|/2 \},$$

where $\Lambda(x)$ is collection of neighbors $x \in V$.

Dynamics is as follows:
$\triangleright$ Turn inactive x active with probability ε_1 when C_t occurs, and with probability $1 - \varepsilon_1$ when C_t^c occurs (=excitation);
$\triangleright$ Turn active x inactive with probability ε_2 when C_t occurs, and with probability $1 - \varepsilon_2$ when C_t^c occurs (=inhibition).

Critical behavior is rich, and shows similarity with critical behavior Ising model.

Integrate and Fire models

Kozma-Puljic (05):
Criticality is arguably key aspect of brains in their rapid adaptation, reconfiguration, high storage capacity, and sensitive response to external stimuli. During recent years, self-organized criticality (SOC) and neural avalanches became important concepts to describe neural systems.

Integrate and Fire model or Abelian sandpile shows self-organised criticality, and macroscopic avalanches at rare times.

Integrate and Fire models

Take graph $G = (V, E)$, where $x \in V$ has degree d_x.

To each vertex $x \in V$, associate height $h_x \in \{0, \ldots, d_x - 1\}$.
Call vertex unstable when $h_x \geq d_x$.
Identify set of sinks.

Dynamics is as follows:
$\triangleright$ Add grain to uniform vertex, and increase h_x by 1;
$\triangleright$ If vertex remains stable, i.e. $h_x < d_x$, do **nothing;**
$\triangleright$ If vertex turns unstable, then **topple:** give grain to each neighbor:
$\triangleright$ When grain is given to sink, it disappears;
$\triangleright$ Repeat until configuration becomes stable.

Inhibitory sandpiles

Inhibition due to presence sinks that inhibit spread electric flow.

Some of these sinks could reflect

response motor networks.

Avalanche sizes display SOC, i.e., probability of avalanche of size k decays as $k^{-\gamma}$.

Macroscopic avalanche might indicate settings where we react to external stimuli, microscopic one information that is damped.

Hebbian learning

In brain, using links strengthens link and makes using link later more likely.

Can be incorporated by adding edge weights. Give grains to neighbors with probability proportional to weight.

Fire together, wire together!

- ▷ Edges with small weights become unimportant;
- ▷ Edges with large weights become important.

Use it or lose it!

Related to pruning. Model is completely unexplored territory.

Hebbian learning

In brain, using links strengthens link and makes using link later more likely.

Can be incorporated by adding edge weights. Give grains to neighbors with probability proportional to weight.

Evolution of weighted graph goes hand in hand with process of integrate and fire on it.

▷ Probability: Call this reinforcement processes.
Simpler versions have attracted substantial attention.
Examples: Reinforced random walk, reinforced graph processes.

Conclusions

▷ Networks useful way to view real-world phenomena:

friendship paradox and centrality.

▷ Unexpected commonality networks:

scale free and small worlds.

▷ Random graph models:

Used to explain properties of real-world networks and as benchmark for brain networks.

▷ Graph theory useful tool for neuroscience.

▷ View brain functionality as stochastic process on brain network.

Material

Lecture notes in preparation:

Random Graphs and Complex Networks

`http://www.win.tue.nl/~rhofstad/NotesRGCN.html`

`http://www.win.tue.nl/~rhofstad/NotesRGCN.pdf`

Lecture notes are aimed at graduate students in mathematics, here I take a much softer approach.

Yet, good source for informal explanations of random graphs for networks, and for basic results.
Non-mathematicians can ignore proofs...